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Quasi-Particle Energy of a Mixture of Two-Component Gas of Bosons

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ABSTRACT

The properties of two component interacting gas of bosons are studied by assuming two slightly different types of interactions between the atoms of the two gases. The system is described by two operators *a* and *b,* that are used to diagonalize the Hamiltonian of the system by the method of Bogoliubov or canonical transformation. The diagonalized Hamiltonian gives the quasi-particle energy spectrum of the system. From the quasi-particle energy spectrum, the role of interaction in each interacting system is studied. The interacting system, which is more likely to be physically acceptable, and can undergo phase transition, is pointed out.

Keywords: Bogoliubov canonical Transformation, Quasi-particle spectrum, interacting system, two-component gas

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1. INTRODUCTION

The fundamentals of the quantum statistics of photons was developed by and Indian scientist, S.N. Bose, in 1924 [1]. German scientist A. Einstein in 1925 [2] used those ideas and predicted the so-called Bose-Einstein Condensation (BEC) in bosonic gases that are

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characterised by integral spin. The most fundamental character of bosons is that any number of bosons can occupy an energy state [3]. The basic idea of BEC is that at some finite temperature, called the critical temperature, the lowest energy state of the assembly of bosons (also called the Zero-Momentum-State, ZMS) is macroscopically occupied. Thus if N is the total number of particles in the assembly, and N_0 particles condense to the ZMS, then $N_0 \simeq N$ but $N_0 < N$ but, and the quantity $\frac{N_0}{N} = \eta$, is called the condensate fraction. Depending on the type of interactions between the particles constituting the boson assembly, N_0 can be depleted, the density of particle number in ZMS can change, and the critical temperature of transition to ZMS changes for an interacting system when compared to the non-interacting system [4, 5].

Occupation of ZMS and condensate fraction for disordered bosons have also been studied [6, 7]

In the last two decades, coherently coupled two component Bose-Einstein-Condensates [8], and Anisotropic pair superfluidity of trapped two-component Bose gases in an optical lattice have been studied [9]. Taking a clue from these studies, we have studied the quasi-particle spectrum of an interacting system of two component mixture of Bose gases. Thus, we consider a gas of bosons which is composed of two different types of boson gases. The number of particles of each gas may be the same or they may slightly differ. This means that the particle density of the two species of bosons may be the same or may slightly differ. Let the two types of bosons be described by two separate operators *a* and *b*, where the operator *a* stands for one type of bosons and operator *b* stands for the second type of bosons. Assume now that the Hamiltonian H for an interacting system of two species of boson gas is given by,

$$H = \varepsilon\left(a^{+}a + b^{+}b\right) + g\left(a^{+}b + b^{+}a\right) \tag{1}$$

where the parameter ε will determine the kinetic energy of the bosons, and g is a parameter that may be a measure of the interaction strength between the particles of the two species.

The terms in Eq. (1) have the following meanings;

(i) $a^{+}a$ refers to the existence of one type of bosons at some position
(ii) $b^{+}b$ refers to the existence of the second type of bosons at the same position as the (i) above
(iii) $a^{+}b$ refers to the creation of one type of boson described by the operator a^{+} and the destruction of the other type of boson described by the operator b.
(iv) Similarly $b^{+}a$ refers to the creation of boson described by the operator b^{+} and destruction of of the other type of boson described by the operator a.

Another system, somewhat similar to the one described by Eq. (1), could have a different form for the Hamiltonian H, such that,

$$H = \varepsilon\left(a^{+}a + b^{+}b\right) + g\left(a^{+}b^{+} + ba\right) \tag{2}$$

Comparing Eq. (1) and Eq. (2), one can see that the first term in each is the same, whereas the second term in one differs from the other. In Eq. (2), the second term exhibits formation of pairs of the two types of bosons constituting the system. This type of Hamiltonian appears in problems such as quantum phase transitions, superfluidity and superconductivity. To obtain the quasi-particle energy spectrum of such a system, the Hamiltonian has to be diagonalized by using what is known as Bogoliubov canonical transformation. Since the Hamiltonians in Eq. (1) and Eq. (2) describe different types of interactions between the bosons of the two-component system, the quasi-particle spectra of the two systems will also be different. This will point out as to how one system differs from the other, and which system is physically more acceptable.

2. THEORY

To diagonalize the Hamiltonians given in Eq. (1) and Eq. (2), we have to define the new operators α and β that will define the Bogoliubov canonical transformation. The new operators α and β that will define the Bogoliubov canonical transformation will be linear combinations of a and b.

For bosons, the Bogoliubov canonical transformation is written as,

$$a = u\alpha - v\beta^{+} \tag{3}$$

$$b = u\beta - v\alpha^{+} \tag{4}$$

The quasi-particle energy spectrum for the Hamiltonians given in Eq. (1) and (2) can be obtained by diagonalizing the Hamiltonians by the Bogoliubov canonical transformation. The difference between the two energy spectra will lead to the understanding of the two energy spectra, and how the physical properties of the two systems will differ from each other.

2. 1. Diagonalization of the Hamiltonian H in Eq. (1)

First, consider the Hamiltonian given by Eq. (1), i.e.

$$H = \varepsilon\left(a^{+}a + b^{+}b\right) + g\left(a^{+}b + b^{+}a\right)$$

Substituting for a, a^{+}, b and b^{+} from Eq.(3) and Eq. (4) in Eq. (1) gives,

$$\begin{aligned} H = {} & \varepsilon\left[u^{2}(\alpha^{+}\alpha + \beta^{+}\beta) - uv(\beta\alpha + \alpha\beta) - uv(\alpha^{+}\beta^{+} + \beta^{+}\alpha^{+}) + v^{2}(\beta\beta^{+} + \alpha\alpha^{+})\right] + \\ & g\left[u^{2}(\alpha^{+}\beta + \beta^{+}\alpha) - uv(\beta\beta + \alpha\alpha) - uv(\alpha^{+}\alpha^{+} + \beta^{+}\beta^{+}) + v^{2}(\beta\alpha^{+} + \alpha\beta^{+})\right] \end{aligned} \tag{5}$$

In Eq. (5), the non-diagonal terms are of the type $(\beta\alpha + \alpha\beta)$, $(\alpha^{+}\beta^{+} + \beta^{+}\alpha^{+})$, $(\alpha^{+}\alpha^{+} + \beta^{+}\beta^{+})$, and hence to diagonalize the Hamiltonian, the proportionality factors of these terms must be put equal to zero. This leads to $uv = 0$ and since $u^{2} - v^{2} = 1$, this means $v = 0$ and $u = 1$. The rest of the Hamiltonian, that is, the diagonalized Hamiltonian can now be written as, say, H_{D1}, i.e,

$$H_{D1} = \varepsilon\left[u^2(\alpha^+\alpha + \beta^+\beta) + v^2(\beta\beta^+ + \alpha\alpha^+)\right] + g\left[u^2(\alpha^+\beta + \beta^+\alpha) + v^2(\beta\alpha^+ + \alpha\beta^+)\right] .. \quad (6)$$

We have to further simplify Eq. (6) by using the values of u and v that must be obtained to diagonalize the Hamiltonian. From the above, we see that $v = 0$ and $u = 1$. Substituting these values of u and v in Eq. (6) gives,

$$H_{D1} = \varepsilon(\alpha^+\alpha + \beta^+\beta) + g(\alpha^+\beta + \beta^+\alpha) \quad (7)$$

Comparing Eq. (1) and Eq.(7) we find that the form of the quasi-particle Hamiltonian is the same as the original Hamiltonian of Eq. (1). In Eq. (7), the new operators α and β replace the old operators a and b. This can be called as identity transformation of the old Hamiltonian into new Hamiltonian. Hence, the quasi-particle energy spectrum will be the same as the original energy spectrum. Thus, such a Hamiltonian cannot lead to any phase transitions.

It can also be verified by using commutation laws for bosons for the two-boson components that the perturbation term $g\left(a^+b + b^+a\right)$ does not commute with the unperturbed term $\varepsilon\left(a^+a + b^+b\right)$, and hence there will be no phase transition.

We now diagonalize the Hamiltonian in Eq. (2). Substituting for a, a^+, b and b^+ from Eq.(3) and Eq. (4) in Eq. (2) , we get,

$$\begin{aligned} H &= \varepsilon\left[u^2(\alpha^+\alpha + \beta^+\beta) - uv(\beta\alpha + \alpha\beta) - uv(\alpha^+\beta^+ + \beta^+\alpha^+) + v^2(\beta\beta^+ + \alpha\alpha^+)\right] + \\ & g\left[(u\alpha^+ - v\beta)(u\beta^+ - v\alpha) + (u\beta - v\alpha^+)(u\alpha - v\beta^+)\right] \\ &= \varepsilon\left[u^2(\alpha^+\alpha + \beta^+\beta) - uv(\beta\alpha + \alpha\beta) - uv(\alpha^+\beta^+ + \beta^+\alpha^+) + v^2(\beta\beta^+ + \alpha\alpha^+)\right] + \\ & g\left[(u^2 + v^2)\alpha^+\beta^+ - 2uv\beta\beta^+ - 2uv\alpha^+\alpha + (u^2 + v^2)\beta\alpha\right] \end{aligned} \quad ..... (8)$$

Using the relevant commutation laws, Eq. (8) leads to,

$$\begin{aligned} H &= 2guv - 2\varepsilon + \left[\varepsilon u^2 + \varepsilon v^2 - 2guv\right]\alpha\alpha^+ + \left[\varepsilon u^2 + \varepsilon v^2 - 2guv\right]\beta\beta^+ + \\ & \left[g(u^2 + v^2) - 2\varepsilon uv\right]\alpha\beta + \left[g(u^2 + v^2) - 2\varepsilon uv\right]\alpha^+\beta^+ \end{aligned} \quad (9)$$

To diagonalize this Hamiltonian in Eq. (9), the coefficients of the non-diagonal terms in Eq. (9) must be put equal to zero. The non-diagonal terms refer to $\alpha\beta$ and $\alpha^+\beta^+$.

Hence

$$g(u^2 + v^2) - 2\varepsilon uv = 0 \quad (10)$$

or

$$\frac{u^2 + v^2}{uv} = \frac{2\varepsilon}{g} \quad (11)$$

The diagonal form of Eq. (9) becomes,

$$H_{D2} = 2guv - 2\varepsilon + \left[\varepsilon(u^2 + v^2) - 2guv\right]\alpha\alpha^+ + \left[\varepsilon(u^2 + v^2) - 2guv\right]\beta\beta^+ \tag{12}$$

Thus the quasi-particle energy eigenvalue E_2 is given by,

$$E_2 = 2guv - 2\varepsilon + \left[\varepsilon(u^2 + v^2) - 2guv\right]2 \tag{13}$$

Substituting for $(u^2 + v^2)$ from Eq. (11) in Eq. (13) gives,

$$E_2 = 2guv - 2\varepsilon + \left[\frac{2\varepsilon^2}{g}uv - 2guv\right]2 \tag{14}$$

or

$$E_2 = \frac{4\varepsilon^2}{g}uv - 2\varepsilon - 2guv \tag{15}$$

The quasi-particle energy eigenvalue E_1 of Eq. (7) is,

$$E_1 = g + \varepsilon \tag{16}$$

Comparing the values of E_1 and E_2, we can point out the exact difference between the two energies. Energy E_1 refers to the difference between the two kinds of energy involved; one being the kinetic energy ε, and the other being the coupling energy g. Whereas in E_2, the energy ε and g are related to each other via the coefficients of transformation u and v. The crux of the problem is as to how to determine the values of u and v. We know that, for bosons,

$$u^2 - v^2 = 1 \tag{17}$$

From Eq. (11) we can write,

$$u^2 + v^2 = \frac{2\varepsilon}{g}uv \tag{18}$$

Adding and subtracting between Eqs. (17) and (18) we get,

$$u^2 = \frac{\varepsilon uv}{g} + \frac{1}{2} \tag{19}$$

$$v^2 = \frac{\varepsilon uv}{g} - \frac{1}{2} \tag{20}$$

Now we can write,

$$u^2v^2 = \frac{\varepsilon^2 u^2 v^2}{g^2} - \frac{1}{4} \text{ or } \frac{1}{4} = \left(\frac{\varepsilon^2}{g^2} - 1\right) u^2 v^2 \text{ or } uv = \frac{g}{2\left(\varepsilon^2 - g^2\right)^{\frac{1}{2}}} \tag{21}$$

Substituting for uv from Eq. (21) in Eq. (15) gives

$$E_2 = \frac{2\varepsilon^2 - g^2}{\sqrt{\varepsilon^2 - g^2}} - 2\varepsilon \tag{22}$$

Eq. (22) shows that the relative magnitudes of the free particle kinetic energy ε and the coupling constant g will determine the magnitude of the quasi-particle energy E_2. For $g = \frac{\varepsilon}{2}$,

$$E_2 = 2.023\varepsilon - 2\varepsilon = 0.023\varepsilon \tag{23}$$

If g is taken as the contact potential between the two bosons (two component boson system) [10], then

$$g = \frac{2\pi a_{BB}}{m_{BB}} \hbar^2 \tag{24}$$

Thus we get,

$$E_2 = 0.023\varepsilon \times 2g = 0.023 \times 2 \frac{2\pi a_{BB}}{m_{BB}} \hbar^2 = (0.023) \frac{4\pi a_{BB} \hbar^2}{m_{BB}} \tag{25}$$

We can now draw the graph for E_2 against a_{BB}

The values of u^2 and v^2 can also be determined as follows;

$$u^2 - v^2 = 1; \frac{u^2 + v^2}{uv} = \frac{2\varepsilon}{g} \tag{26}$$

$$u^2 = 1 + v^2 \text{ and } u = \sqrt{1 + v^2} \tag{27}$$

Substituting for u^2 in Eq. (26) we get

$$\frac{u^2 + v^2}{uv} = \frac{2\varepsilon}{g} = \frac{1 + v^2 + v^2}{\sqrt{1 + v^2}.v} = \frac{1 + 2v^2}{v\sqrt{1 + v^2}}$$

$$\text{or} \quad 2\varepsilon v\sqrt{1 + v^2} = g(1 + 2v^2)$$

$$\text{or}\quad 4\varepsilon^2 v^2\left(1+v^2\right)=g^2(1+2v^2)^2=g^2(1+4v^4+4v^2)$$

$$\text{or}\quad 4\varepsilon^2 v^2+4\varepsilon^2 v^4=g^2+4g^2v^4+4g^2v^2$$

$$\text{or}\quad \left(4\varepsilon^2-4g^2\right)v^4+\left(4\varepsilon^2-4g^2\right)v^2-g^2=0 \tag{28)}$$

Let $v^2=x$, then

$$\left(4\varepsilon^2-4g^2\right)x^2+\left(4\varepsilon^2-4g^2\right)x-g^2=0 \tag{29}$$

$$x=\frac{-(4\varepsilon^2-4g^2)\pm\sqrt{(4\varepsilon^2-4g^2)^2+4(4\varepsilon^2-4g^2)g^2}}{2(4\varepsilon^2-4g^2)}$$

$$=\frac{-(4\varepsilon^2-4g^2)\pm(4\varepsilon^2-4g^2)^{\frac{1}{2}}\sqrt{(4\varepsilon^2-4g^2)+4g^2}}{2(4\varepsilon^2-4g^2)}$$

$$v^2=\left(-\frac{1}{2}\right)\pm\frac{\varepsilon}{\left(4\varepsilon^2-4g^2\right)^{\frac{1}{2}}} \tag{30}$$

Thus v^2 has two values,

$$v^2=-\frac{1}{2}-\frac{\varepsilon}{\left(4\varepsilon^2-4g^2\right)^{\frac{1}{2}}}\quad\text{and}\quad v^2=-\frac{1}{2}+\frac{\varepsilon}{\left(4\varepsilon^2-4g^2\right)^{\frac{1}{2}}} \tag{31}$$

The first value of v^2 will lead to imaginary v, and this is not acceptable since v is real. Thus,

$$v^2=-\frac{1}{2}+\frac{\varepsilon}{\left(4\varepsilon^2-4g^2\right)^{\frac{1}{2}}} \tag{32}$$

Now for v to be real the term,

$$\frac{\varepsilon}{\left(4\varepsilon^2-4g^2\right)^{\frac{1}{2}}}>\frac{1}{2}$$

$$\text{or}\quad \varepsilon>\frac{1}{2}\left(4\varepsilon^2-4g^2\right)^{\frac{1}{2}}$$

$$\text{or}\quad \varepsilon^2>\frac{1}{4}\left(4\varepsilon^2-4g^2\right)$$

$$\text{or}\quad \varepsilon^2>\left(\varepsilon^2-g^2\right) \tag{33}$$

which is correct since $\varepsilon^2 > \left(\varepsilon^2 - g^2\right)$ provided $g < \varepsilon$. In fact, combining Eq. (20) and Eq.(21) we get the same equation as in Eq. (32). Similarly we can calculate the value of u^2.

2. 2. Demixing of two-component Boson superfluid

The transition temperature T_c for the Bose-Einstein-Condensation is given by,

$$T_c = \frac{3.31\hbar^2 n^{\frac{2}{3}}}{4\pi^2 kM} \tag{34}$$

where n is the critical density $\frac{N}{V}$ (particle number density). Temperature T_c is the critical temperature at which the addition of more particles leads to BEC or the superfluid state. Here M is the particle mass of the given boson gas. Now for a two-component mixture of bosons, let m_1 be the atomic mass of one component, and m_2 be the atomic mass of the second component of the mixture of bosons. The critical particle number density n may be the same or different for each component. For ideal mixture (non-interacting components) of the two components, Eq.(34) shows that T_c may be different for each component. T_{c1} for large m_1 component will be smaller compared to the T_{c2} for small m_2. This means that if $T_c \le T_{c1}$, the two component Bose system will be superfluid. However, if $T_c > T_{c1}$, then the component with mass m_2 alone will be superfluid. This may be called de-mixing of the two-component boson superfluid by changing T_c.

2. 3. Calculations

Different combinations of two component bosonic mixtures can be used to calculate E_2 provided we know the values of a_{BB}. The following three specific combinations have been used [11].

Table 1. Values of scattering lengths a_{BB} for bosonic Isotopes.

ISOTOPES	SCATTERING LENGTHS (a_{BB}) a_0 = Bohr Radius
${}^{39}_{19}K + {}^{39}_{19}K$	$(140^{+3}_{-6})a_0$ (Same Bosons)
${}^{41}_{19}K + {}^{41}_{19}K$	$(85 \pm 2)a_0$ (Same Bosons)
${}^{39}_{19}K + {}^{41}_{19}K$	$(113 \pm 3)a_0$ (Two-component Bosons)

Using the values of a_{BB} given in Table 1, the values of quasi-particle energy E_2 in Eq. (25) are calculated. The values are given in Table 2.

Table 2. Calculated Values of Quasi-Particle energy for bosonic Isotopes.

ISOTOPES	SCATTERING LENGTHS (a_{BB}) in terms of Bohr Radius (a_0)	Quasi-Particle Energy Density (E_2) in Joules.
${}^{41}_{19}K + {}^{41}_{19}K$	$83a_0$	$4.116\times10^{-52}\,J$
${}^{41}_{19}K + {}^{41}_{19}K$	$85a_0$	$4.215\times10^{-52}\,J$
${}^{41}_{19}K + {}^{41}_{19}K$	$87a_0$	$4.314\times10^{-52}\,J$
${}^{39}_{19}K + {}^{41}_{19}K$	$110a_0$	$5.594\times10^{-52}\,J$
${}^{39}_{19}K + {}^{41}_{19}K$	$113a_0$	$5.747\times10^{-52}\,J$
${}^{39}_{19}K + {}^{41}_{19}K$	$116a_0$	$5.899\times10^{-52}\,J$
${}^{39}_{19}K + {}^{39}_{19}K$	$134a_0$	$6.985\times10^{-52}\,J$
${}^{39}_{19}K + {}^{39}_{19}K$	$140a_0$	$7.298\times10^{-52}\,J$
${}^{39}_{19}K + {}^{39}_{19}K$	$143a_0$	$7.456\times10^{-52}\,J$

3. DISCUSSION

In many-body physics, whether it is nuclear physics, condensed matter physics or solid-state physics, a modern theme in research is to study the effect of inter-particle interactions among the particles constituting the system. There is still no method of deriving from first principles as to what is the nature of two-body or many-body interactions among the particles constituting the system. A number of interactions, whether attractive or repulsive, that were proposed from time to time were based on the intuition or guesswork of the brilliant scientists who proposed them.

In the recent past, there has been a tremendous increase in activity in the field of cold atoms where lasers are used to confine the particles. Both experimental [12] and theoretical [13] studies on the properties of confined cold atoms are available in these articles. Over the past more than a decade, there has been an increased interest in the so-called optical lattice, along with the ability to tune all the relevant interactions, including changing from weak to strong interactions, attractive (negative scattering length) to repulsive (positive scattering length) interactions via the method of Fesh-back resonances [14, 15]. Early studies were mainly confined to one-component cold-atom systems with finite-range interactions. Recently multi-component systems, such as binary bosonic mixtures in deep optical lattices have been investigated [16-19]. To explore quantum many-body physics, two component lattice bosons [20] and coupled superfluidity of binary Bose mixtures have been studied [21] to understand how the intercomponent and intracomponent interactions, the attractive and or repulsive interactions and the strong and weak interactions, determine the properties of the ultra-cold two component Bose gases. In this study, we have considered a two-component Bose gas in which the interaction between particle of the two-components Bose gases is assumed to be some constant. It is rather a contact potential between the particles of two-component Bose gas. We have thereby, calculated the quasi-particle energy eigenvalues for different two-component Bose gases having different scattering lengths that were obtained from experimental observations.

The calculations in this manuscript given in Table 2 show that the quasi-particle energy of the two-component boson systems in the superfluid state (ZMS) is more or less the same in magnitude for any different two-component boson mixture. As a rule in the superfluid state, the magnitude of the energy should not depend on the internal physical characteristics of the components constituting the two-component boson mixture. The inherent property of the superfluid is frictionless flow without dissipation. Thus, the superfluid state should not remember the physical properties of the constituents composing the mixture.

It is possible that under certain conditions, there could be demixing of the Bose components or there may be phase separation. If the number of bosons of one component are much more than the other, then at some critical temperature T_c, one component may be superfluid and the other may not be, and the superfluid system may be surrounded by the normal system. Such a system will be studied theoretically separately later.

The physics of two-component bosonic mixtures when in a degenerate state plays important role in understanding the properties of ultra-cold atomic gases of different species [22]. Two-component Bose droplets and mixtures have been studied experimentally to understand whether superfluidity can sustain, and how the superfluid behavior in each component is affected by the presence of the other [23-27]. However, the effect of finite temperature and the effect of variation in the boson-boson scattering length (a_{BB}) on the state of coupling of the two components and their superfluid behavior needs to be studied. The interspecies interaction can result in low-energy phonon excitations of the two components [28]. Our results suggest that low energy phonon excitations can lead to coupled superfluidity in both the components. Since the transition temperature to superfluidity will be different for each component, coupled superfluidity may exist only if the temperature T of the mixture is lower than the lower transition temperature of the two components. This can be explained by using Eq. (34).

Now if the temperature of the mixture is such that one component is not in the superfluid state, then there could exist a dissipative drag on the superfluid component by the non-superfluid component. But close to zero but finite temperature, such a situation may not arise since at such very low temperatures there will exist long-wavelength (very low energy) phonons, and the phonon energy fluctuations will be so small that dissipative drag may not exist; hence superfluidity of the two- component Bose mixture will sustain [21].

4. CONCLUSIONS

The quasi-particle energy excitation spectrum (E_2), varies as the boson-boson scattering length (a_{BB}) varies (the value of E_2 increases as the value of a_{BB} increases). This is natural since at large scattering length, to maintain coherence in the superfluid state, the energy (E_2) must increase with increase in a_{BB}. The value of E_2 is quite small and this refers to the phonon spectrum of the two-component boson system. Such a system is homogeneous and stable since there is no singularity in the value of E_2. It will be interesting to study the conditions under which the de-mixing of the components of the boson-boson mixture can take place. We plan to study this problem in a later communication.

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