PROPERTIES OF A MIXTURE OF BOSONS AND FERMIONS AT ZERO KELVIN TEMPERATURE

By

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DECLARATION

DECLARATION BY CANDIDATE

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This work is dedicated to almighty God and to my family.

ABSTRACT

Quantum materials reveal unexpected and exotic behavour when subjected to extreme conditions such as low temperature and/or high pressure. Ultra cold gases provide a very powerful tool for simulation and study of condensed matter systems. Based on recent developments on applications of quantum gases, we have to look for experimental models that can be used to probe and manipulate particles in quantum state and look at their theoretical framework in order to understand their properties. In this work, a mixture of bosons and fermions at zero Kelvin temperature is considered and its properties studied. Most theoretical work have been devoted to a system of two Bose condensates. It is in this research therefore, that we consider a system of Bose condensate with fermionic impurities and look at the properties that arise due to their interaction. The aim of this research was to determine the density distribution of bosons and fermions that are trapped in isotropic external potential and compare their density distribution for different values of ratio of their interaction strength h/g. Gross-Pitaevski mean field equation for the boson distribution in the trap is solved by utilizing Thomas Fermi Approximation to extract the density profile of the fermions and bosons components. The results show that the Fermi gas will constitute a shell around a core of Bose condensate for $h > g$ and it forms a core inside the Bose condensate for $h > g$. For $h = g$, both states exist simultaneously, the fermions has a constant spatial density where the bosons are localized. In this work, the existence of three distinct states of the system under variation of the ratio of the interaction strength h/g has been confirmed.

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a_O	Bohr radius 5.292×10^{-11}m
a_{BB}	Boson-Boson scattering length
a_{BF}	Boson-Fermion scattering length
a_B	Boson-boson s-wave scattering length
a_{ho}	Harmonic oscillator length
E_F	Fermi energy
g	Boson-boson interaction strength/Coupling constant
h	Boson-fermion interaction strength/Coupling constant
$\hbar$	(Planck constant $h/2\pi$) 1.0545×10^{-34}Js
k_B	Boltzmann constant 1.381×10^{-23}J/K
m_O	Atomic mass unit 1.661×10^{-27}kg
m_B	Mass of the Bosonic atom
m_F	Mass of the Fermionic atom
m	Reduced mass $m_B m_F/(m_B+m_F)$
N_F	Number of fermions
n_B	Density of the Bosonic atoms
n_F	Density of the Fermionic atoms
p	Momentum
p_F	Fermi momentum
R_F	Radius of the Fermionic cloud
R_B	Radius of the condensate
r	Distance from the trap center.
T_c	Critical temperature
μ	Chemical potential
ω	Angular frequency
x, y, z	Position vector
λ	Aspect ratio
BCS	Bardeen Cooper Schrieffer
BEC	Bose-Einstein Condensate
MOT	Magneto-Optical Trapping

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CHAPTER ONE

INTRODUCTION

1.1 Background information

All particles, elementary particles such as electrons as well as composite particles such as atoms and molecules, belong to one of two possible classes: they are either fermions or bosons. The class in which a particle belongs to is determined by its spin. If the spin is an odd multiple of $\hbar$ /2, the particle is a fermion. For even multiplesof $\hbar$ /2, it is a boson. Examples of fermions are electrons or ${}^{6}_{3}$Li atoms.${}^{7}_{3}$Li and photons are examples of bosons. The quantum properties of a particle are influenced by its bosonic or fermionic nature. For a system of identical particles, the many particle wave function must be symmetric under the exchange of two particles for bosons and anti-symmetric under the exchange of two particles for fermions. A direct consequence of this (anti-) symmetrization postulate is that it is impossible for two fermions to occupy the same quantum state, as illustrated in Figure1.1. This is called the Pauli Exclusion Principle. No process can add a fermion to an already occupied state. The process is inhibited by Pauli blocking. For bosons it is favorable to occupy the same state, as illustrated in Figure 1.1 and the more bosons that are already in this state, the higher the probability that another boson is transferred to it. This property is called Bose enhancement.

Bose enhancement leads to a phase transition at high phase-space densities, corresponding to low temperatures and/or high densities. When the phase-space density is increased past a certain critical value, the occupation of the ground-state rapidly becomes macroscopic. This effect is called Bose-Einstein Condensation (BEC)

(Inguscio, *et. al*., 1999). Bosons behave very differently from fermions at ultra-low temperatures, where the atoms behave more like waves than as point-like particles.

When identical bosons are cooled to near absolute zero (about –0.1833 K), they coexist in the same energy state called the Zero Momentum State (ZMS). Their movements fall into step at a single low energy level, and they behave as one unified "super-atom," called a BEC. Lately, a lot of attention has also been given to fermions especially in view of the possibility of achieving temperatures low enough to observe a BCS type transition. Cooling fermions is harder than cooling bosons.

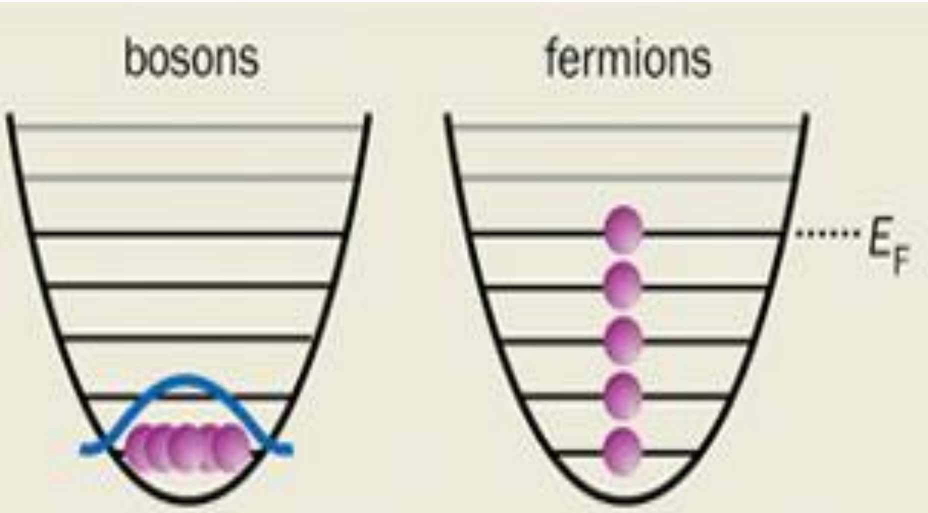

Figure 1.1: At absolute zero, gaseous boson atoms all end up in low energy state. The fermions in contrast, fill the available states with one atom per state-shown here for one dimensional harmonic confining potential.

The main difficulty arises from Fermi statistics as the s-wave collisions between spin polarized fermions in a magnetic trap are forbidden by Pauli's Exclusive principle. A common strategy uses sympathetic cooling, which is based on s-wave collision between fermions and a second gaseous component made either of fermions in a different internal state or bosons. The latter choice seems to minimize the effects of Pauli blocking. This leads finally to two trapped condensates.

The developments in the trapping and cooling of atoms have made it possible to investigate the properties of dilute gases at very low temperature where bosonic or fermionic character of the atom are scrutinized. When laser cooling is used, very cold Bose Condensate can be obtained in the temperature range of nano- Kelvin (10^{-9}K). A mixture of Bose Condensate and Fermi gas was studied assuming that a degenerate Fermi gas (Degenerate Fermi gas is a system at very low temperature where classical laws or classical statistical mechanics is not applicable) interacts with a Bose Condensate (Roati, *et. al.*, 2002). The mixture was assumed to be trapped in an external potential $V_{ext}(r)$. The atoms interact by elastic collisions. Because of extremely low temperature, the atoms have very low kinetic energies, and this permits replacement of their short range interaction potential by a delta function.

The conditions in which the bosons and fermions can significantly overlap was determined and also how the bosons and fermions distribution in space can be strongly modified by various components such as the number of bosons and fermions and the values of g, h, ω and a (distance to the trap center or the oscillator ground state width). In the mixture of bosons and fermions the following patterns can exist:

i. Bosons and fermions can significantly overlap,
ii. Bosons surround the fermions and the fermions constitute the core,
iii. Fermions surround the bosons and the bosons constitute the core.

All the above possibilities were studied in detail in the present work.

1.2 Statement of the Problem

A number of experiments have been conducted on systems with two condensates. Most theoretical work concerning multi component condensates have been devoted to a system of two Bose condensates. Theoretical framework for a system of a Bose condensate with fermionic impurities has not been fully developed despite the fact that cooling of fermions has already been reported (DeMarco, et *al*, 1999)

In this work, the properties of a degenerate Fermi gas which is interacting with a Bose condensate was considered. The miscibility properties of boson-fermion mixture at zero temperature Kelvin in a trap is discussed in detail. This has been done within the Thomas- Fermi approximation, where the separation of the components is studied numerically as a function of the inter-particle interaction. An analytical study of the miscibility of fermions-boson mixtures in the uniform case has been carried out in order to obtain the types of phase boundaries that may occur in this system at zero Kelvin temperature, in order to determine its properties.

1.3 Objective

1.3.1 General Objective

The aim of this research was to obtain the density distribution of bosons and fermions that are trapped in isotropic external potential and compare their density distribution for different values of ratio of their interaction strength h/g.

1.3.2 Specific Objective

The specific objectives are;

i. Determine the density profiles for bosons and fermions trapped in isotropic external potential as a function of r, the distance from the trap centre.
ii. Compare the density distribution for the trapped bosons and fermions from trap center for different values of boson-boson interaction strength h and boson-fermion interaction strength, g.

1.4 Justification of the study

One of the first practical applications suggested for the ultra-cold Bose- Fermi gases is in quantum computing, which is based on the theory of quantum mechanics. Quantum computing will be able to quickly solve problems that would be too complex for today's digital computers.

Unlike Newtonian mechanics, which describes the motions of stars and planets in space, quantum mechanics predicts the movements of electrons and the other invisible particles that make up all matter. And unlike digital computing, which relies on only two possible information states-on or off, quantum computing depends on the idea that in the quantum world, both states could exist at the same time, superimposed on top of each other.

In addition to quantum computing, researchers are interested in using ultra-cold polar molecules as models for studying complicated physical systems, such as superconductors. They can also be used in precision measurement of fundamental physical constants. For doing precision measurement, the particles have to be ultra-cold so that they can be held on to for a long time. At ultra-cold temperatures, particles move so slowly that they can be trapped more easily, and they won't have shifts caused by their motion.

Based on these applications, there is need to understand quantum mechanics more fully. To do this, we have to design experiments that can be used to probe and manipulate particles in a quantum state. One such model is an ultra-cold highly condensed atom gas called a Bose-Einstein Condensate (BEC). There is need also to study their properties, such as their position and momentum in order to understand them. This work therefore was aimed at obtaining their density distribution and look at how their density distribution is altered by inter-particle interaction strengths.

1.5 Scope

A simple theoretical analysis of the situation in which a Bose condensate and a degenerate fermi gas coexist will be considered. The atoms are assumed to be trapped in the same external harmonic oscillator potential which is isotropic. ^{40}K and ^{87}Rb Fermi-Bose mixture at zero temperature Kelvin have been chosen for this study. The reason for choosing ^{40}K and ^{87}Rb atom is the fact that experiments on these atoms have been carried out (Inguscio, *et. al.,* 1999, Mølmer, K., *et. al.*, 1998, Dalfovo, F., *et. al.,* 1999, Butts, D. A. and Rokhsar, D.,1997, Roati, G., 2002, Pitaevskii, L. and Stringari, S., 2003).

CHAPTER TWO:

THEORY AND LITERATURE REVIEW

2.1 Introduction

Since the experimental realization of Bose- Einstein condensation in dilute gases of Rubidium (Inguscio, *et. al.,* 1999, Mølmer, K., *et. al.*, 1998, Dalfovo, F., *et. al.,* 1999, Butts, D. A., and Rokhsar, D., 1997), Sodium (Roati, G., 2002, Pitaevskii, L. and Stringari, S.,2003), lithium (Pethick, C. J., et. al., 2001), and hydrogen (DeMarco, B. and Jin, D. S., 1999), a great deal of interest in Bose Condensed systems have concentrated on the topic of multi- component condensates. This field was stimulated by the successful demonstration of overlapping condensates in different spin states of rubidium in a magnetic trap (De Marco, B., et. al., 1999, Granade, S. R., et. al., 2002) and of sodium in an optical trap (Schreck, F., *et. al*, 2001), the (binary) mixtures being produced either by sympathetic cooling, which involves one species being cooled to below the transition temperature only through thermal contact with an already condensed Bose gas, or by radiative transitions out of a single component condensate. Since then a host of experiments has been conducted on systems with two condensates, exploring both the dynamics of component separation (Truscott, G., et. al., 2001), and measuring the relative quantum phase of the two Bose- Einstein condensates (Hadzibabic, Z., *et al*, 2002). Most of the theoretical work concerning multi-component condensates (Roati, G., *et. al*., 2002, Modugno, G., *et al*, 2002, Nygaard, N. and Molmer, K., 1999, Roth, R., *et. al*., 2002, Miyakawa, T., *et. al*., 2000, Griffin, A., Albus, A.P., *et. al*., 2002, 1996, Viverit, L. and Giorgini, 2002, Albus, A.P, 2003 and

Ospelkaus, C., *et. al*., 2006) has been devoted to systems of two Bose condensates. However, other systems are of fundamental interest, one of these being a Bose condensate with fermionic impurities, for instance a system of ^{40}K -^{87}Rb Fermi-Bose mixture. In particular the possibility of sympathetic cooling of fermionic isotopes has been predicted in both ^{6}Li-^{7}Li (DeMarco, B., et. al., 1999) ^{39}K-^{40}K, and ^{41}K-^{40}K (Modugno, M., *et al*, 2003). Magneto - optical trapping of the fermionic potassium isotope ^{40}K has been reported (Fedichev, P. O., *et. al*., 1996).

2.2 Atomic fermi gases

Let me start by reviewing the properties of a trapped non-interacting Fermi gas. I then describe the procedure to produce a degenerate Fermi gas of ^{40}K atoms. In particular, the procedure to bring fermions into degeneracy exploits the technique of sympahetic cooling in which a bosonic gas of ^{87}Rb atoms acts as a refrigerator. ^{40}K-^{87}Rb Fermi-Bose mixture is an extremely rich system which gives us a twofold possibility: on the one hand, it can be compared directly with the behavior of two atomic gases obeying to two different quantum statistics and, on the other hand, interspecies interaction effects can be investigated. In particular, our Fermi-Bose mixture exhibits a large interspecies attraction which strongly affects both the density distribution and the dynamics of the system.

2.2.1 Trapped Fermi gas

While the bosonic degeneracy involves the formation of a Bose-Einstein condensate, the fermionic degeneracy leads to a single particle occupation of quantum states. At zero temperature, the occupation number of each fermionic quantum state is equal to one, up to energies close to the Fermi energy E_F, and is zero for larger energies. The Fermi energy E_F corresponds to the higher energy level occupied at T=0 K, and sets the relevant energy scale of the system. This tight packing creates a Fermi sea of particles where a minimum size is maintained by the so-called Fermi pressure (Ferlaino, F., 2004). Furthermore, added particles cannot penetrate into the Fermi Sea and this gives rise to the Pauli blocking of collisions. It is interesting to note that all these features arise some-how from the properties of symmetry of the fermionic wave-function. In particular, a system composed by N identical fermions is described by a wave-function which is anti-symmetric under the interchange of any pair of particle coordinates. On the contrary, a bosonic function is completely symmetric. This fundamental difference leads to different statistical mechanics which governs these two classes of particles. A Fermi gas is described by Fermi-Dirac distribution (Ferlaino, F., 2004)

$$f(\mathrm{r},\mathrm{p}) = \frac{1}{\mathrm{e}^{\beta[\mathrm{H}(\mathrm{r},\mathrm{p})-\mu]}+1} \qquad 2.1$$

Where β is 1/K_BT with K_B the Boltzmann constant. The function $f(r,p)$is the occupation probability of a state of energy ε, where $H(r,p)\psi(r,p) = \varepsilon\,\psi(r,p)$. The chemical potential μ fixes the number of atoms in the gas. The zero-temperature Fermi distribution is equal to one for energies lower than E_F =μ (T = 0K) and zero otherwise. The presence of a finite temperature smoothes the step-wise transition from one to zero occupation numbers. In particular, at a finite temperature a shell of amplitude k_BT opens around E_F, and the unitary occupation is no more guaranteed. For such a distribution, only fermions with energies in a shell near the Fermi surface provide a response of the system to external perturbations (Ferlaino, F., 2004). Thus, the ratio between the temperature T and the Fermi temperature, T_F=E_F/K_B, defines the degree of degeneration of the system. Note that the scenario is opposite in a Bose-condensed system where all the particles participate in the response (Dalfovo, F., *et. al.*, 1999).

The main quantities involved in a Fermi gas which is confined by a harmonic trap with a cylindrical symmetry, such as temperature and the number of atoms, depend also on the trapping potential. We now briefly describe the basic features of a harmonically trapped Fermi gas. A more detailed description can be found in (Butts, D. A., and Rokhsar, D, 1997, Roati, G., 2002).

The Hamiltonian H(r, p) of a harmonically trapped Fermi gas is known to be

$$H(r,p) = \frac{p^2}{2m} + V_F(r), \tag{2.2}$$

Where $m,$ is the atomic mass of the fermion and V_F(r) is the harmonic trap potential. Our harmonic potential exhibits a cylindrical symmetry along the z-axis, also named axial direction. The trapping frequencies are $(\omega_1, \omega_2, \omega_3) \equiv (\omega_r, \omega_r, \omega_z)$with $\omega_{r,z}$ the radial and axial frequency, respectively. Introducing the aspect ratio of the trap $\lambda = \omega_z/\omega_r$, V_F is given by

$$V_F(r) = \frac{1}{2} M\omega_r^2 (x^2+y^2+\lambda z^2). \tag{2.3}$$

The single particle levels are the Eigen values of the harmonic oscillator. If the thermal energy far exceeds the level spacing ($k_BT >> \hbar\, \omega_r$), we can replace the discrete single-particle harmonic spectrum with a continuum one, whose density of energy states is (Butts, D. A., and Rokhsar, D., 1997)

$$g(\epsilon) = \frac{\epsilon^2}{2\lambda(\hbar\omega_r)^3} \tag{2.4}$$

The chemical potential μ is then given by the normalization condition for the total number of fermions N_F in the trap

$$N_F = \frac{1}{(2\pi\hbar)^3}\int \frac{g(\epsilon)\partial\epsilon}{e^{\beta(\epsilon-\mu)}} \tag{2.5}$$

Eq. (2.5) also fixes the Fermi energy E_F of the system (Butts, D. A., and Rokhsar, D., 1997). Solving the integral, one indeed finds

$$E_F = \hbar\omega_r[6\lambda N_F]^{\frac{1}{3}} \tag{2.6}$$

which sets the relevant energy scale of the system. From the Fermi energy, we can define the typical size of a trapped degenerate gas

$$E_F = m\omega_r^2 R_F^2 \;\; \rightarrow \;\; R_F = \sqrt{\frac{E_F}{m\omega_r^2}} \; = a_{ho}(48N_F)^{\frac{1}{6}} \tag{2.7}$$

Where the harmonic oscillator length is $a_{ho} = \sqrt{\hbar/m}\,\omega_{\,ho}$ and $\omega_{\,ho} = (\omega_r^2\omega^z)^{1/3.}$

From Eq. (2.7), it follows that if the number of fermions N_F>>1, the size of the trapped Fermi cloud is much greater than a_{ho} (Ferlaino, F., 2004). This is as a result of blocking by the Pauli Exclusion Principle. This effective repulsion between fermions in the trap is known as Fermi pressure, and leads to a bigger size of the cloud with respect to the harmonic oscillator length a_{ho}. This is another important difference with respect to both a "classical" gas and a Bose condensed gas. Indeed decreasing the temperature, the size of a classical gas continuously shrinks accordingly to the classical Boltzmann distribution as shown in Figure 2.1. The size of a non-interacting Bose-Einstein Condensate is instead temperature-independent and at T=0 K it is exactly equal to a_{h0} because they all occupy the lowest state of the harmonic oscillator. If one also introduces the two-body repulsive interaction between condensed atoms, the radius of the cloud also increases with N_B. In particular, the radius of the condensate R_B scales with N_B as $N_B^{1/5}$ (Ferlaino, F., 2004) which is slightly different from the behavior found for a Fermi gas, $R_F \propto N_F^{1/6}$ (Butts, D. A., and Rokhsar, D 1997)

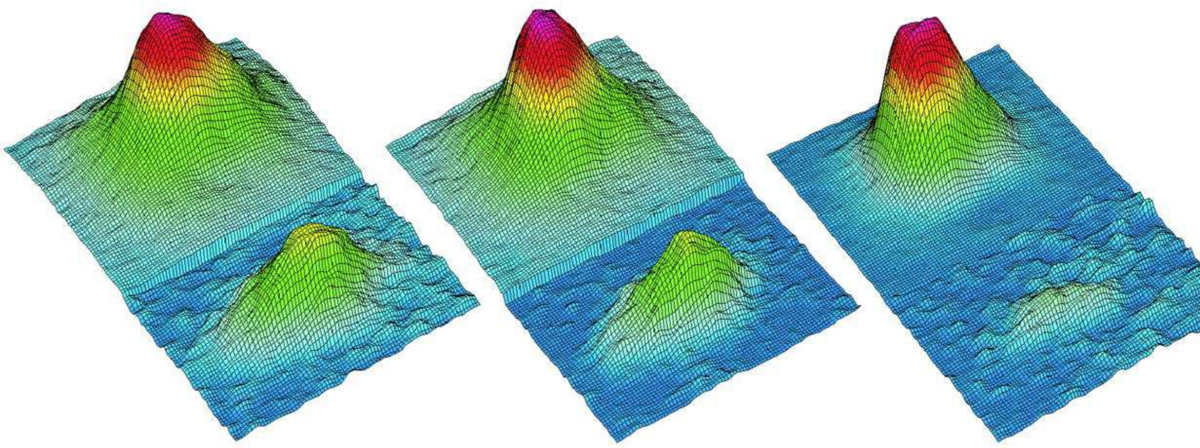

Figure 2.1: False-color reconstruction of the density distributions of a gas with fermionic ^{40}K (front) and bosonic ^{87}Rb (back) during the evaporative cooling process, as detected after a ballistic expansion of the mixture (Griffin, A., 1996).

2.2.2 Spatial and momentum distribution.

For a temperature different from zero, the density distribution of a degenerate Fermi gas has to be calculated numerically by integrating the distribution function (equation 2.1) in the momentum space. At T= 0K, one instead finds an analytic expression (Ferlaino, F., 2004):

$$n(r, T = 0) = \frac{8\lambda N}{m\omega_r^2}\left[1 - \frac{\rho^2}{R_F^2}\right]^{\frac{3}{2}} \tag{2.8}$$

Where ρ is the effective distance $\rho = (x^2+y^2+\lambda^2 z^2)^{1/2}$ defined for a harmonic trap with a cylindrical symmetry. Another important quantity is the momentum distribution of the cloud. Indeed, in the experiments, most of the information about the sample is obtained looking to the absorption signal of the cloud, after the sudden release from the trap. When the confinement is switched off, the cloud performs a rapid and adiabatic expansion and we are able to measure the velocity distribution by imaging the atoms. The momentum distribution (Thomas-Fermi distribution) at zero temperature is

$$n(p, T = 0) = \frac{1}{(2\pi)^3} \int d^3\, r\Theta(p_F(r) - P) \tag{2.9}$$

Where $\Theta(p_F(r) - P)$is the unit step function and the Fermi momentum $p_F = \sqrt{2mE_F}$ The integral (2.9) gives (Dalfovo, F., et. al., 1999)

$$n(p, T = 0) = \frac{8N}{\pi^2 p_F^3} \left[1 - \frac{p^2}{p_F^2}\right]^{\frac{3}{2}} \tag{2.10}$$

The momentum distribution of the degenerate Fermi gas is isotropic i.e. the momentum distribution depends only on the magnitude of p. As we will discuss in the next section, this is an important difference with respect to the case of a Bose gas.

2.2.3 Comparison with the Bose-Einstein condensate

A Bose gas shows a behavior somehow opposite with respect to the one exhibited by Fermi gas. The difference between these two systems arises entirely from their different statistical mechanical nature. Above all, trapped bosons undergo a phase transition as the critical temperature T_c is reached and all the atoms prefer to occupy macroscopically a single state. Furthermore, differently from a gas of identical fermions, the condensed bosons collide with each other. Due to the low temperatures ($T \leq T_c \approx 100$ nK) and the diluteness of the cloud, the inter-particle interaction can be described in a simple way. Indeed one can consider that each boson experiences a mean field potential produced by all the other particles on the gas (Dalfovo, F., et. al., 1999, Pitaevskii, L., et. al., 2003). This approximation is somehow justified by the fact that, at low temperature, just two-body collisions survive. The interatomic potential can be written as a δ-function using the method of the pseudo-potentials (Ferlaino, F., 2004),

$$V(r'-r) = g\delta(r'-r) \tag{2.11}$$

The coupling constant g, at the first order of the perturbation, takes the form

$$g = \frac{4\pi\hbar^2 a_{BB}}{m} \tag{2.12}$$

Where a_{BB} is the boson-boson s-wave scattering length. This interaction introduces in the system a sort of rigidity which yields to phenomena of superfluidity observed in Bose-Einstein condensate (Dalfovo, F., et. al., 1999). For a large number of atom N_B, the interaction energy is notably larger than the kinetic energy. In this limit, one can neglect the latter contribution to the energy and the system is known to be in the Thomas-Fermi regime (Ferlaino, F., 2004). At T=0 K, the density distribution of a trapped condensate has an inverted-parabola shape

$$n_B(r) = \frac{R_B^2}{2g}\left[1 - \frac{\rho^2}{R_B^2}\right] \tag{2.13}$$

Where R_B is the maximum radius of the cloud (Dalfovo, F., et. al., 1999 Butts, D. A., and Rokhsar, D., 1997)

$$R_B = \left(\frac{15\lambda g N_B}{4\pi}\right)^{\frac{1}{5}} \tag{2.14}$$

The typical energy scale of a Bose gas is the zero-temperature chemical potential μ which scales with the atom number more rapidly than the Fermi energy ($\mu \propto N_B^{2/5}$ while $E_F \propto N_B^{1/3}$). Another important difference is connected to the spatial and momentum distribution of the two clouds. Even if the two gases exhibit a similar spatial distribution, their momentum distributions differ in a profound way. The momentum distribution of both a thermal cloud and a Fermi gas turns out to be isotropic. On the contrary, in a condensate, $n_B(p)$is anisotropic in a asymmetric trap due to the non-linearity of the inter-particle interaction. Furthermore, the widths of the momentum distribution scale in the opposite way: p_F increases with N_F, while the typical momentum for a condensed atom decrease with particle number, since $p_B \alpha \frac{1}{R_B}$ due to the Heisenberg uncertainty principle.

2.2.4 The cooling a Fermi gas

A Fermi gas of ^{40}K atoms well below T_F is produced using the technique of sympathetic cooling with ^{87}Rb atoms (Ferlaino, F., 2004). In this section, we give a rapid overview on the experimental technique used to produce atomic Fermi gas. A detailed description can be found in the PhD thesis of Giacomo Roati (Roati, G., 2002).Since the first achievement of Bose-Einstein condensation, the standard technique to cool an atomic gas below the temperature of degeneracy consists of a pre-cooling phase based on laser cooling which carries the system at T ≈100μK and of an evaporative cooling phase (Pethick, C. J., and Smith, H., 2001). The initial cooling phase for both alkali bosons and fermions proceeds via laser cooling and MOT. The sub-sequent cooling phase has instead to be different for the two species. Bosons can indeed exploit techniques based on re-thermalization, while fermions cannot collide down to μK (DeMarco, B., et. al., 1999). In particular, bosons are transferred from the MOT into a magnetic trap, where a forced evaporative cooling is applied to bring the gas into degeneracy. The evaporation of bosons is performed usually by using a radio-frequency signal which removes selectively the hottest atoms from the trap. The key requirement for the usefulness of this technique is clearly the existence of a large elastic collisional rate between atoms which allows for an efficient thermalization of the gas. In general, the elastic cross-section depends on the temperature. At very low temperature, the only significant contribution to the collisional rate is given by the s-wave scattering amplitude which is temperature-independent. The other contributes (p-wave, d-wave, etc) are proportional to the temperature and thus suppressed down to 100μK.

The situation is even more complicated for identical fermions because inter-atomic collisions are completely suppressed in such a system. As a consequence, the evaporative cooling fails for spin-polarized fermions and another cooling procedure has to be found. One can circumvent this problem using some form of mutual or sympathetic cooling between two types of distinguishable particles, either two spin states of the same atomic species or of two kinds of atoms. In the first scheme, fermions are simultaneously trapped in two different spin states and evaporating cooling is then performed on both components (De Marco, B., et. al., 1999, Granade, S. R., et. al., 2002). Thermalization is now assured by s-wave collisions between these two spin states. The other scheme exploits the idea of to mix fermions with a gas of bosons which can be efficiently cooled using the usual evaporative cooling. The Fermi gas decreases its temperature by colliding with bosons which act like a refrigerator. This latter technique is known as sympathetic cooling and has been carried out with success at ENS (Paris) (Schreck, F., *et. al,* 2001), at Rice (Texas) (Truscott, G., et. al., 2001) and at MIT (Hadzibabic, Z., *et al*, 2002).

2.3 Quantum degeneracy of boson-fermion mixture

Quantum degeneracy was first reached with mixtures of bosonic ${}^{7}_{3}\mathrm{Li}$ and fermionic ${}^{6}_{3}\mathrm{Li}$ (Truscott, G., et. al., 2001), and also with ${}^{23}_{11}\mathrm{Na}$ (boson) and ${}^{6}_{3}\mathrm{Li}$ (fermion) as well as ${}^{87}\mathrm{Rb}$ (boson) and ${}^{40}\mathrm{K}$ (fermion) at ultra-low temperatures. These boson-fermion mixtures offer unique possibilities to study the effects of quantum statistics directly. Recently, superfluidity has been obtained experimentally in a mixture of Bose condensed gas and superfluid Fermi gas of two Lithium atoms, ${}^{6}\mathrm{Li}$ and ${}^{7}\mathrm{Li}$ (Ferrier-Barbut, I., *et al*, 2014), where a new mechanism of superfluidity and its instability was observed. In a Bose-Fermi superfluid mixture, there could be a fully mixed phase and a fully separated phase, and a third phase could consist of pure fermions in equilibrium with a mixture of bosons and fermions (Tylutki, M., et. al., 2016) or it could be pure bosons in equilibrium with a mixture of bosons and fermions. Ultimately the conditions for stability of homogenous phase of the mixture have to be studied. The phase diagram of a weakly interacting Bose-Fermi mixture at zero temperature is derived starting from the following expression (Tylutki, M., et. al., 2016) for energy density $E(n_F;n_B)$, i.e,

$$E(n_F;n_B)=\frac{1}{2}g_{BB}n_B^2+g_{BF}n_Bn_F+\frac{3}{5}E_Fn_F \qquad (2.15)$$

Where n_B is the density of bosons, n_F is the density of fermions and $E_F=\frac{\hbar^2k_F^2}{2m_F}$, where k_F is the fermi momentum and m_F is the mass of the fermi particles; g_{BB} is the bosonic inter-species coupling constant and g_{BF} is the boson-fermion coupling constant. These coupling constants are related to corresponding scattering lengths a_{BB} and a_{BF}. Recent experiments have shown that Bose-Fermi scattering length does not depend on the internal state of the Fermi atoms (Delehaye, M., et al, 2015). The stability condition predicted by the energy density of equation (2.15) for the uniform mixture is

$$n_F^{1/3} = \frac{\hbar^2 g_{BB}}{2m_F g_{BF}^2} \tag{2.16}$$

For n_F larger than this critical value, the uniform mixture is unstable and the system exhibits either partial or full phase separation (Tylutki, M., et. al., 2016). However interacting superfluid fermions and the Bose-condensed bosons are coupled via the interspecies interaction term determined by g_{BF}. In general, the number of bosons and the number of fermions are not equal in the mixture. But the stability conditions of the mixture will depend on the relative number of bosons and fermions in the mixture, and this will also determine the phase separation, i.e., whether the box is filled with pure fermions, while bosons are still in the mixed phase in the remaining volume; and this corresponds to the partially mixed phase of the mixture (Tylutki, M., et. al., 2016).

Bosonic lasers have been developed based on BEC of exciton-polaritons in semiconductor micro cavities. These electrically neutral bosons coexist with charged electrons and holes, which are thus Bose-Fermi mixtures. In the presence of magnetic fields, the charged particles are bound to their cyclotron orbits, while the neutral exciton-polaritons move freely. In this way the magnetic fields dramatically affect the phase diagram of a mixed Bose-Fermi mixture, switching between fermionic lasing, incoherent emission and bosonic lasing (Vladimir, P., et.al, 2016).

2.4 Thermodynamics of a mixture of bosons and fermions

There have been a number of experimental observations (Schreck, F., *et. al,* 2001, Truscott, G., et. al., 2001, Hadzibabic, Z., *et al*, 2002) and theoretical calculations (Khanna, K.M., et. al., 2003) on the ultra-cold trapped Bose Fermi (BF) mixtures of alkali metal atoms (Albus, A.P., et. al., 2002, Viverit, L., and Giorgini, 2002).

Consequently, a number of theoretical investigations were done to study the static property (Mølmer, K., 1998), phase diagram and phase separation (Albus, A.P, 2003), stability conditions (Ospelkaus, C., et. al., 2006). Interaction-driven Dynamics and collective excitation (DeMarco, B., et. al., 1999) of the trapped BF mixtures. These investigations were mainly done in the limit of the temperature tending to zero or at T=0, and studied the Bose condensate of the bosonic atoms.

The interest in this attempt is to study the properties of binary BF mixtures, and to address the question of how BF interactions affect the thermodynamics properties of such mixtures. The objective is to calculate the condensate fraction and the critical temperature of BEC at various interaction strengths. Particularly the effect of BF attractive interaction and BF repulsive interaction as studied on the variation of condensate fraction and the critical temperature.

The theory developed (Modugno, M., *et al*, 2003) to study the in-homogenous interacting Bose gas will be used to study the thermodynamic properties of the BF mixtures. The modification of the theory in the presence of the BF interaction will be pointed out wherever necessary.

Now the trapped dilute mixture is considered as a system in a thermodynamic equilibrium under the grand canonical ensemble whose thermodynamic variables are N_B is the total number of trapped bosonic atoms, N_F is the total number of trapped fermionic atoms, T is the absolute temperature, μ_B is the chemical potential of the bosons and μ_F is the chemical potential of the fermions. The Hamiltonian of the system is written as (in units of $\hbar = 1$) (Sakwa, T.W., *et al*, 2013)

$$H = H_B + H_F + H_{BF} \tag{2.17}$$

Where

$$H_B = a_B^{+}\left[-\frac{\nabla^2}{2m_B} + V_{trap}^{B} - \mu_B\right]a_B + \frac{g}{2}a_B^{+}a_B^{+}a_B a_B \tag{2.18}$$

$$H_F = a_F^{+}\left[-\frac{\nabla^2}{2m_F} + V_{trap}^{F} - \mu_F\right]a_F \tag{2.19}$$

$$H_{BF} = h a_B^{+}a_B a_F^{+}a_F \tag{2.20}$$

Where a_B is the Bose field operator that annihilates a boson at the position r, and a_F the Fermi field operator that annihilates a fermion at the position r, H_B stands for the Hamiltonian for bosons that interact with each other, the interaction strength between two bosons is given by g, H_F is the Hamiltonian for fermions which are assumed not to interact with each other, and H_{BF} is the Hamiltonian for the interaction between bosons and fermions and h is the interaction strength between bosons and fermions. For spherically symmetric systems, the trap potentials are;

For bosons,

$$V_{trap}^{B}(r) = \frac{m_B\omega_B^2 r^2}{2} \tag{2.21}$$

And for fermions,

$$V_{trap}^{F}(r) = \frac{m_F\omega_F^2 r^2}{2} \tag{2.22}$$

Where m_B and ω_B are the bosonic mass and the trap frequency; and m_F and ω_F are the corresponding values for fermions. The interaction between the bosons, and bosons and fermions are described by contact potentials and are given by coupling constants g and h respectively

$$g = \frac{4\pi\hbar^2 a_{BB}}{m_B} \tag{2.23}$$

$$h = \frac{2\pi\hbar^2 a_{BF}}{m_r} \tag{2.24}$$

Where a_{BB} is the scattering length for boson-boson interaction, a_{BF} is the scattering length for boson fermion interaction and m_r, is the reduced mass and is given by

$$m_r = \frac{m_B m_F}{m_B + m_F}$$

The fermion- fermion interaction is neglected since the s-wave collisions are forbidden by Pauli Principle. In the dilute system the Hamiltonian describing the BF coupling, can be treated in a self consistent mean field manner such that,

$$H_{BF} = h[a_B^+ a_B < a_F^+ a_F > + < a_B^+ a_B > a_F^+ a_F - < a_B^+ a_B >< a_F^+ a_F >] \tag{2.25}$$

This kind of decomposition has been used extensively for theoretical investigations of BF mixtures at zero temperature (Inouye, S., *et al*, 1998, Fedichev, P. O., et. al., 1996). The Bose field operator can now be decomposed into a C-number part plus an operator with vanishing expectation value; that is,

$$a_B(r,t) = \phi(r)e^{-i(\epsilon_0 - \mu_B)t} + \bar{a}_B(r) \tag{2.26}$$

Where $\phi(r)$ represents the condensate wave function with eigen values ϵ_0 and $\bar{a}_B(r)$ is an operator that represent the excitations of the condensate. Now the equation of motion for $a_B(r,t)$ can be written as,

$$i\frac{\partial a_B(r,t)}{\partial t} = \left[-\frac{\nabla^2}{2m_B} + V_{trap}^B - \mu_B\right] a_B(r,t) + g a_B^+(r,t) a_B(r,t) a_B(r,t)$$
$$+ h a_F^+(r,t) a_F(r,t) a_B(r,t) \tag{2.27}$$

From equation (2.26), we can write the local density of the condensate as

$$n_C(r) = |\phi(r)|^2 \tag{2.28}$$

And the depletion $\bar{n}(r)$ as,

$$\bar{n}(r) =< \bar{a}_B^{+}(r,t)\bar{a}_B(r,t) > \tag{2.29}$$

And the density of the Fermi gas, $n_F(r)$ is the well known relation;

$$n_F(r) =< a_F^{+}(r,t)a_F(r,t) > \tag{2.30}$$

Equation (2.27) can be transformed by using the Bogoliubov transformation given by

$$\bar{a}_B(r,t) = \sum_i \left[u_i(r)\alpha_i e^{-i\epsilon_i t} + V_i^{\star}\alpha_i^{+} e^{i\epsilon_i t}\right] \tag{2.31}$$

The transformation of equation (2.27) will lead to an equation that will define a quasi particle excitation energy relative to the condensate eigen value (given in equation (2.26) and the quasi particle amplitude are u_i and v_i. The thermal number of quasi particle is given by,

$$< \alpha_i^{+}\alpha_i >= \frac{1}{(ze^{\beta\epsilon_i} - 1)} \tag{2.32}$$

Where,

$$z = e^{\beta(\epsilon_0 - \mu)} = 1 + \frac{1}{N_C} \tag{2.33}$$

ϵ_i is the energy of the ith state and$N_C = \frac{1}{e^{\beta(\epsilon_0-\mu)}-1}$ is the number of Bose particles in the condensed state.

The density of depletion $\bar{n}(r)$ is given by

$$\bar{n}(r) = \frac{|u_i(r)|^2 + |v_i(r)|^2}{ze^{\beta\epsilon_i} - 1} + |v_i(r)|^2 \tag{2.34}$$

And

$$\bar{n}(\epsilon, r) = \frac{m_B^{3/2}}{\sqrt{2}\pi^2}\left[\frac{1}{(ze^{\beta\epsilon} - 1)} + \frac{1}{2} - \frac{\epsilon}{2\epsilon_{HF}}\right]\left[\epsilon_{HF} - V_{trap}^{B}(r) + \epsilon_0 - 2gn_B(r) - hn_F(r)\right]^{\frac{1}{2}} \tag{2.35}$$

For $\epsilon_i = \epsilon_0$, the energy of the condensed state.

$$\epsilon_{hF} = [\epsilon^2 + g^2 n_C^2(r)]^{\frac{1}{2}} \tag{2.36}$$

The total density of the Bose gas is,

$$n_B(r) = n_C(r) + \bar{n}(r) \tag{2.37}$$

The local density $n_F(r)$of the Fermi gas will be found by writing

$$a_F(r,t) = \sum_i a_{Fi} c_i e^{-i\epsilon_i t} \tag{2.38}$$

Where $a_F(r,t)$ is written in terms of new Fermi operator c_i that annihilates a fermions at the $i$thstate such that,

$$n_{Fi} = |a_{Fi} r|^2 < c_i^+ c_i > \tag{2.39}$$

And thus

$$n_F(\epsilon, r) = \frac{m_F^{3/2}}{\pi^2\sqrt{2}} \frac{1}{e^{\beta(\epsilon-\mu_F)} + 1} \left[\epsilon - V_{trap}^F(r) - h n_B(r)\right]^{1/2} \tag{2.40}$$

Where the Fermi distribution is given by

$$< c_i^+ c_i >= \frac{1}{e^{\beta(\epsilon-\mu_F)} + 1} \tag{2.41}$$

Now the general expression that will be used for the condensate fraction η is

$$\eta = \frac{N_c(r)}{N_B} = \left[1 - \left(\frac{T}{T_c^0}\right)^3\right] \tag{2.42}$$

The value of the condensate fraction η_c near the critical temperatureT_c is given by

$$\eta_c = \frac{N_c^0}{N_B} = \left[1 - \left(\frac{T}{T_c^0}\right)^3\right] - 2.1825\left(\frac{T}{T_c^0}\right)^2 \frac{1}{(N_B)^{\frac{1}{3}}} \tag{2.43}$$

Where $KT_c^0 = 0.94\hbar\omega_B N_B^{\frac{1}{3}}$

2.5 Ultracold mixtures of bose and fermi gases

BEC of the trapped ultra-cold atomic gases has been observed experimentally (Truscott, G., et. al., 2001), and a lot of progress has been made in the study of Fermi-degenerate systems and Bose-Fermi mixtures (Schreck, F., *et. al*, 2001, Hadzibabic, Z., *et al*, 2002, and Albus, A.P., et. al., 2002). Recently, the Feshbach-resonance method (this is the application of an external magnetic field to change the sign of the scattering length from negative to positive or positive to negative) has been used for molecular formations for two fermions (Viverit, L. and Giorgini, 2002, Mølmer, K., 1998), and two bosons (Albus, A.P, 2003, Ospelkaus, C., et. al., 2006). Interaction-driven Dynamics and ^{87}Rb-^{40}K boson-fermion hetero-nuclear molecules (Schreck, F., *et. al*, 2001, DeMarco, B., et. al., 1999). Effects of inter-particle correlations have been studied on boson-fermion systems (Hadzibabic, Z., *et al*, 2002).

Another theory used for the study of the Bose-Fermi mixtures is the Quasi-chemical equilibrium theory that was originally developed to study the electron system in superconductors (Modugno, M., *et al*, 2003 Fedichev, P. O., et. al., 1996), has been used to study the behavior of boson-fermion, boson-boson and fermion-fermion mixtures of ultracold atomic gases. In this theory inter-particle interactions can easily be included to obtain equilibrium structures, particularly, the inter-particle interactions for the s- wave scattering process.

2.5.1 Introduction to quasi-chemical equilibrium theory

Let us consider an atomic- gas mixture composed of two atomic species A_1 and A_2 with masses m_1 and m_2. The gas A_1 may be boson and A_2 may be fermion, such that the gases are described in quantum statistics, Bose-Einstein statistics for bosons and Fermi Dirac statistics for fermions. The mixture can have three kinds of combinations; boson-boson (BB), fermion- fermion (FF) and boson fermion (BF)

In the quasi-chemical equilibrium theory (Bruun, G. M. et. al., 1998), the molecular formation or dissociation process in the mixture is written as

$$A_1 + A_2 \leftrightarrow (A_1A_2) = M \tag{2.44}$$

Where M is the composite molecule with mass m_M which is bosonic in the BB and FF mixtures, and is fermionic in the BF mixture

The mass defectΔm_M of the molecule is defined as,

$$\Delta m_M \equiv (m_M - m_1 - m_2) \tag{2.45}$$

For the bound molecule, $\Delta m_M < 0$, and the molecule is stable in both vacuum and gases, and the molecule has a molecular binding energy $\Delta E = \Delta m_M c^2$, c is the velocity of light in a vacuum. If $\Delta m_M > 0$, we get what we call the resonance state, and such a state is unstable in a vacuum, but may exist as a stable state in gases. Thus both bound molecule state and resonance states could be considered.

If μ_1 is the chemical potential of specific gas 1, $\mu_2$the chemical potential of gas 2 and μ_M is the chemical potential of the molecule composed of A_1 and A_2, then the equilibrium condition for the process in equation (2.44) is given by

$$\mu_1 + \mu_2 + \mu_M = \Delta E_M \tag{2.46}$$

The molecular binding energy ΔE_M in equation (2.46) is very small $10^{-(5\text{-}10)}$ eV in molecular formation in the ultra-cold atomic gases. This could be treated as a measure of interaction energy between the two atoms constituting the molecule in the mixture.

For free uniform gases, the particle densities are given by the Bose and Fermi statistics (Bruun, G. M. and Burnett, K., 1998)

For bosonic atoms,

$$n_\alpha = \frac{1}{(2\pi)^3}\int \frac{d^3r}{[e^{(\epsilon_\alpha-\mu_\alpha)/K_BT}-1]} + n_\alpha^{(0)}\frac{1}{\left(\lambda_{T,\alpha}\right)^3}B_{\frac{3}{2}}\left(-\frac{\mu_\alpha}{K_BT}\right) + n_\alpha^{(0)} \qquad (2.47)$$

And for fermionic atoms,

$$n_\alpha = \frac{1}{(2\pi)^3}\int \frac{d^3r}{[e^{(\epsilon_\alpha-\mu_\alpha)/K_BT}+1]} \equiv \frac{1}{\left(\lambda_{T,\alpha}\right)^3}F_{\frac{3}{2}}\left(-\frac{\mu_\alpha}{K_BT}\right) \qquad (2.48)$$

Where K_B is the Boltzmann is constant, $\lambda_{T,\alpha}$ is the thermal de Broglie length of the particle A_α at temperature T, and the Bose $B_A(\nu)$ is;

$$B_A(\nu) = \frac{1}{\mathrm{T}(A)}\int_0^\infty \frac{x^{A-1}dx}{e^{x+\nu}-1} \qquad (2.49)$$

Where $\mathrm{T}(A)$is the gamma function and ν is fugacity= $e^{\frac{\mu}{K_BT}}$

The Fermi- Dirac function $F_A(\nu)$ is;

$$F_A(\nu) = \frac{1}{\mathrm{T}(A)}\int_0^\infty \frac{x^{A-1}dx}{e^{x+\nu}+1} \qquad (2.50)$$

For T >T_c, where T_c is the critical temperature at which some transition can take place, in equation (2.47) the first part gives the thermal part of the number density , and the second part gives the condensed part of the number density. These can be written as

$$n_\alpha^{(th)} = \frac{(m_\alpha)^{3/2}}{2\pi} \xi \left(\frac{3}{2}\right) \tag{2.51}$$

$$n_\alpha^{(0)} = n_\alpha \left[1 - \left(\frac{T}{T_c}\right)^{\frac{3}{2}}\right] \tag{2.52}$$

Where $\xi\left(\frac{3}{2}\right)$ is the Riemann Zeta function

2.5.2 Molecular formation in the BF, FF, and BB mixtures

We consider the BF mixture (A_1=B, A_2= F) with molecular formation process B+F↔M = (BF), the atom or the molecule densities n_B, n_F and n_m in the atom molecule equilibrium are obtained from the equilibrium condition given in equation (2.46) along with the values given in equation (2.47) and (2.48) under different conditions involving the values of ΔE_M and temperature T, mixed state of atoms and molecules become stable in the sense of equilibrium (Bruun, G. M. and Burnett, K., 1998).The atom molecule equilibrium in the BF mixture can be explained by the competition between the quantum statistical effect and the binding energy of M. The molecule state gives a free energy reduction because of $\Delta E_M > 0$ but at low T, the molecules constitute the Fermi Dirac states which have a large kinetic energy due to the occupied high energy one particle state. However in the dissociated state, the bosons can reduce the kinetic energy largely as they condense into the BEC at low T. Thus depending upon the positive or the negative values of ΔE_M , the dissociated or the molecular state becomes stable, and a mixed state appears then.

In the FF mixtures, (A_1=F_1, A_2=F_2) the atom molecule equilibrium is considered through a process F_1+F_2 ↔M = (F_1F_2) in the same way as in the BF mixture. In the crossover theory (Modugno, M., *et al*, 2003 Fedichev, P. O., et. al., 1996 Stoof, H. T. C., et. al., 1996)), two kinds of bare fermions become dressed quasifermion and quasi-molecule state (or Cooper pair state) appear as physical degree of freedom. The change in strength of the attractive interaction between bare fermions gives the crossover between the BCS states (weak interaction) and the molecular BEC states (strong interaction).

The atom-molecule equilibrium in the BB mixture is considered through the process B_1+B_2 ↔M= (B_1B_2), and the equilibrium condition is again given by equation (2.46). If all the bosons condense into the BEC, then the chemical potentials μ_{B1}, μ_{B2} and μ_M are zero at T=0. If however, the chemical potentials μ_{B1} and μ_{B2} are negative and $\Delta E_M < 0$ at T=0, then $\mu_M = 0$ (the BEC of M). But if $\Delta E_M > 0$ (dissociated phase), then the values of the chemical potentials $\mu_{B1} = \mu_{B2} = 0$ (the BEC's of B_1 and B_2) and thus $\mu_M = -\Delta E_M$ and there will be no molecules or $n_M = 0$. This is a brief description of how the quasi-chemical theory can be used to study the properties of BB, FF and BF mixtures. More details can be found in (Bruun, G. M. and Burnett, K., 1998).

CHAPTER THREE:

THEORETICAL DERIVATIONS AND METHODOLOGY

3.1 Introduction

This chapter consists of three parts. In the first part brief discussion on the trapping and cooling processes has been discussed. In the second part, theoretical derivations have been done with the aim of obtaining an expression which can be used to extract the density profiles for fermionic and bosonic components of the mixture. This was done by solving Gross-Pitaevskii equation by utilizing Thomas Fermi Approximation. Calculations using the derived expressions were carried out in the last part and the results tabulated.

3.2 Methodology

In this work, a mixture of bosons and fermions at 0K temperature is considered and its properties studied. Gross- Pitaevski mean field equation for the boson distribution in the trap is solved by utilizing Thomas Fermi Approximation. TFA exploits the fact that at very low temperature, kinetic energy of the particle is so low that its kinetic energy operator can be neglected in the Gross- Pitaevski equation for boson. The solution to the Gross- Pitaevski equation gave functions that were used to determine the density distribution for bosons and fermions as a function of distance from the trap center. The results for density distribution for different values of boson-boson interaction strength h and boson-fermion interaction strength g were tabulated and Ms Excel was used extract the density profile of the fermions and bosons components.

3.3 Trapping and cooling processes; Feshbach resonance and laser cooling

The trapping of the boson-fermion mixture is via the Feshbach resonance method in which the spin dependence of the inter-atomic interaction gives rise to both open and closed channels. In the context of ultra-cold gases, they are of special importance as they allow the modification of the interactions between the atoms, in particular the scattering length (Chin, C., et. al., 2010). A good example of such a mixture is ^{6}Li-^{40}K and ^{87}Rb-^{40}K (Wille, E., *et al*, 2008). More details about collisions among the atoms of gases can be found in (Walraven, J.T.M., 2013). It is also known that when the ^{87}Rb (boson) atoms are not completely evaporated, various regimes of mixtures are accessible, ranging from dense thermal ^{87}Rb cloud of 10^7, ^{87}Rb right at the phase transition point interacting with a moderately degenerate Fermi gas (^{40}K) of 2×10^6 atoms to deeply degenerate mixtures with almost pure condensate (Ospelkaus, C., et al., 2006). Stability conditions for Bose-Fermi mixtures have been studied leading to the values of N_B and N_F for stability (Modugno, M., *et al*, 2003).

There is experimental realization of Bose-Fermi superfluid mixtures of dilute ultra-cold atomic gases (Inguscio, *et. al.,* 1999, Mølmer, K., *et. al.*, 1998, Dalfovo, F., *et. al.*, 1999, Butts, D. A., *et. al.,* 1997). Depending upon the values of the scattering lengths, and the amount of bosons and fermions, a uniform Bose-Fermi mixture could exhibit a fully mixed phase, or a fully separated phase, or a pure fermionic phase co-existing with a mixed phase.

In ultra-cold atomic gases, the strength of the interspecies and intra-species interaction can be varied by means of an external magnetic field (what is called Feshbach resonance method). This leads to the exploration of the whole phase diagram of the mixture (Castin, Y., et. al., 2015, Delehaye, M., et al, 2015). The Bose-Fermi mixture could be of two types. One in which the bosonic superfluid is the minority component, and the second in which the fermions are the minority component (Karpiuk, T., *et.al*, 2004). The miscibility and immiscibility is determined by interaction. It is also found that the Bose-Fermi phase diagram is known to admit, in addition to a fully mixed phase and a fully separated phase, also a third phase consisting of a pure fermions in equilibrium with a mixture of fermions and bosons (Ludwig, D., *et.al*, 2011). Laser cooling can, lead to very low temperatures, of the order of 10^{-9}K (nano Kelvin). At these temperatures the Fermi gas will be degenerated, Bose gas will be condensate, and the two systems can interact. There could be intergas and intragas interactions. However, the intergas interaction in a Fermi gas could be neglected due to Pauli Principle and the interactions between bosons, and bosons and fermions have to be taken into account. The boson-boson interaction is represented by $g$and the boson fermion interaction is represented by h. The strength of both the interactions must be proportional to the S-wave scattering lengths.

3.3 Theoretical Derivations

Assuming that the degenerate gas interacts with Bose condensate and the mixture is trapped in an external potential$V_{ext}(r)$, the atoms will interact by elastic collision. At low temperatures, the atoms will have low kinetic energies and thus permit replacement of their short range interaction with a delta function. In the mean field description of the single particle, wave function $\psi(r)$, assumed to describe all bosons in the gas is governed by Gross Pitaevskii Equation.

$$\left\{-\frac{\hbar^2}{2M}\nabla^2 + V_{ext}(r) + gN_B|\psi(r)|^2\right\}|\psi(r)| = \mu\psi(r) \tag{3.1}$$

The Thomas Fermi Approximation exploits the fact that at low temperature, kinetic energy of the atom is so small that the kinetic energy operator $(-\frac{\hbar^2}{2M}\nabla^2 r)$ can be neglected. Hence equation 3.1 becomes (Nygaard, N., and Molmer, K., 1999),

$$(V_{ext}(r) + gN_B|\psi(r)|^2)\psi(r) = \mu\psi(r) \tag{3.2}$$

Dividing equation 3.2 by $\psi(r)$ yields,

$$V_{ext}(r) + gN_B|\psi(r)|^2 = \mu \tag{3.3}$$

Re-arranging equation 3.3 gives,

$$n_B(r) = N_B|\psi(r)|^2 = \frac{\mu - V_{ext}(r)}{g} \tag{3.4}$$

Where $N_B|\psi(r)|^2 = n_B(r)$ is bosonic density, $V_{ext}(r)$is the external confining potential and μ is the bosonic chemical potential (energy per particle). The value of μ is fixed by normalization condition. $\int d^3r.n_B(r) = N_B$on the boson density $n_B(r) = N_B|\psi(r)|^2$ so that it yields the total number of particles. In a harmonic oscillator potential,

$$V_{ext}(r) = \frac{1}{2}\mathrm{M}\omega^2\mathrm{r}^2 \tag{3.5}$$

where r is the distance from the trap centre.

and μ is determined analytically to be

$$\mu = \left[\frac{15}{8\pi}Ng\left(\frac{m\omega^2}{2}\right)^{3/2}\right]^{2/5} \tag{3.6}$$

When N is very small, Thomas Fermi Approximation gives a good approximation to the exact distribution of particles and to the single particle energy. For low kinetic energy, short range interaction potential is replaced by a delta function of strength g and h. g and h represent the boson-boson and boson-fermion interaction strength proportional to the respective s-wave scattering lengths.

For the particles in the TFA, due to Pauli's exclusion principle, the atoms in the degenerate gas of fermions do not occupy a single state. Hence there is no equivalent of Gross Pitaevskii equation for fermions. Instead, the particles will be described by classical position and momenta. However, we use the quantum mechanical result that a volume in phase space d^3rd^3k can accommodate, $\frac{d^3rd^3k}{\hbar^3(2\pi)^3}$ fermions, i.e., if the local density $n_F(r)$ will have the wave numbers within the integral $0 \leq k \leq k_F(r)$. The fermions will experience a local potential **(**Nygaard, N., and Molmer, K., 1999),

$$V(r) = V_{ext}(r) + \mathrm{h}.n_F(r) \tag{3.7}$$

For the particles in motion for such a potential, it is possible to define a local Fermi vector $k_F(r)$by,

$$\mathrm{E_F} = \frac{\hbar^2 \mathrm{k_F(r)}^2}{2\mathrm{M}} + \mathrm{V(r)} \tag{3.8}$$

So that the volume of the local Fermi sea in k space is

$$\frac{4}{3}\pi k_F(r) = (2\pi)^3 n_F(r) \tag{3.9}$$

Local Fermi vector will be given by

$$k_F(r) = \left(6\pi^2 n_F(r)\right)^{1/3} \tag{3.10}$$

In low temperature limit, p-wave scattering can be neglected. The suppression of the s-wave scattering amplitude due to antisymmetry of the many body function implies that the spin polarized fermions may constitute a non-interacting gas; hence the energy density of fermionic component is given by the expression;

$$\frac{\hbar^2 k_F^2(r)}{2M} + V_{ext}(r) + h.n_B(r) = E_F \tag{3.11}$$

Re-writing equation 3.11 using equation 3.4 and equation 3.10 yields,

$$E_F = \frac{\hbar^2\left(6\pi^2 n_F(r)\right)^{2/3}}{2M} + V_{ext}(r) + \frac{h(\mu - V_{ext}(r))}{g}$$

$$E_F = \frac{\hbar^2\left(6\pi^2 n_F(r)\right)^{2/3}}{2M} + V_{ext}(r) + \frac{h}{g}\mu - \frac{h}{g}V_{ext}(r)$$

$$E_F = \frac{\hbar^2\left(6\pi^2 n_F(r)\right)^{2/3}}{2M} + V_{ext}(r) - \frac{h}{g}V_{ext}(r) + \frac{h}{g}\mu$$

$$E_F = \frac{\hbar^2\left(6\pi^2 n_F(r)\right)^{2/3}}{2M} + \left[1 - \frac{h}{g}\right]V_{ext}(r) + \frac{h}{g}\mu \qquad (3.12$$

In isotropic traps, the trapping potential $V_{ext}(r)$ by bosonic component is equal to the local potential experienced by the fermions.

$$V_{ext}(r) = V(r)$$

$$\mu - g.n_B(r) = V_{ext}(r) + \mathrm{h}.n_F(r)$$

$$\mu = V_{ext}(r) + g.n_B(r) + \mathrm{h}.n_F(r) \qquad (3.13)$$

The TFA for both components solves the coupled equation 3.12 and 3.13.

The mean occupation number of a single particle energy states with energy ε_n is given by

$$f(\varepsilon_n) = \frac{1}{\zeta^{-1}\varepsilon^{\beta\varepsilon_n} + a} \qquad (3.14)$$

Where $\zeta = \varepsilon^{\beta\mu}$, is the fugacity, $\beta = \frac{1}{KT}$ and

$$a = \begin{cases} -1 & \text{Bose Einstein Statistic} \\ +1 & \text{Fermi Dirac Statistic} \\ 0 & \text{Maxwell} - \text{Boltzmann Distribution} \end{cases}$$

In Fermi Dirac, the mean occupation number can become utmost one (Pauli's exclusion principle). Hence equation 3.14 becomes,

$$f(\varepsilon_n) = \frac{1}{\varepsilon^{-\beta\mu}\varepsilon^{\beta\varepsilon_n} + 1}$$

$$f(\varepsilon_n) = \frac{1}{\varepsilon^{\beta(\varepsilon_n - \mu)+1}}$$

$$f(\varepsilon_n) = \frac{1}{\varepsilon^{\frac{(\varepsilon_n - \mu}{KT})} + 1} \qquad (3.15)$$

For harmonically trapped gases, density of states as a function of energy is given by (Butts, D. A., et. al., 1997),

$$g(\varepsilon) = \frac{\varepsilon^2}{2(\hbar\omega)^3} \tag{3.16}$$

The number of particles in the excited states can be calculated according to

$$N_F = \int_0^\infty f(\varepsilon)g(\varepsilon).d\varepsilon \tag{3.17}$$

Integrating equation 3.17 with $f(\varepsilon) = \begin{cases} 1 & \varepsilon > E_F \\ 0 & \varepsilon < E_F \end{cases}$

Gives,

$$N_F = \int_0^{E_F} g(\varepsilon).d\varepsilon \tag{3.18}$$

Substituting equation (3.16) in equation (3.18) gives,

$$N_F = \int_0^{E_F} g\frac{\varepsilon^2}{2(\hbar\omega)^3}.d\varepsilon \tag{3.19}$$

Integrating equation (3.19) yields,

$$N_F = \frac{E_F^3}{6(\hbar\omega)^3} \tag{3.20}$$

The fermionic energy will be given by,

$$E_F = (6N_F)^{\frac{1}{3}}\hbar\omega \tag{3.21}$$

Combining equation 3.12 and 3.21 we get,

$$(6N_F)^{\frac{1}{3}}\hbar\omega = \frac{\hbar^2\left(6\pi^2 n_F(r)\right)^{2/3}}{2M} + \left[1 - \frac{h}{g}\right]V_{ext}(r) + \frac{h}{g}\mu \tag{3.22}$$

Combining equation 3.5, 3.6 and 3.22 yields,

$$(6N_F)^{\frac{1}{3}}\hbar\omega = \frac{\hbar^2\left(6\pi^2 n_F(r)\right)^{2/3}}{2M} + \left[1 - \frac{h}{g}\right]\frac{1}{2}\mathrm{M}\omega^2\mathrm{r}^2 + \frac{h}{g}\left[\frac{15}{8\pi}Ng\left(\frac{m\omega^2}{2}\right)^{3/2}\right]^{2/5} \quad (3.23)$$

Re-arranging equation 3.23 gives,

$$n_F(r) = \frac{\left\{\frac{2M}{\hbar^2}\left[(6N_F)^{\frac{1}{3}}\hbar\omega - \left(1 - \frac{h}{g}\right)\frac{1}{2}\mathrm{M}\omega^2\mathrm{r}^2 - \frac{h}{g}\frac{15}{8\pi}N_B g\left(\frac{m\omega^2}{2}\right)^{3/2}\right]^{\frac{2}{5}}\right\}^{\frac{3}{2}}}{6\pi^2} \quad (3.24)$$

Equation 3.24 gives an expression for fermionic density.

The first part of equation (3.13 was used to get the bosonic density

$$V_{ext}(r) + g.n_B(r) + h.n_F(r) = \mu$$

$$g.n_B(r) = \mu - V_{ext}(r) - h.n_F(r)$$

$$n_B(r) = \frac{\mu - V_{ext}(r) - h.n_F(r)}{g} \quad (3.25)$$

The strength of the boson-boson interaction g is chosen to give maximal overlap between the two atomic clouds. In order to have clouds of comparable sizes, we equate the Thomas Fermi expression for the radius of the Bose Condensate $(15N_B g/4\pi M\omega^2)^{\frac{1}{5}}$ and the radius of zero temperature Fermi gas $(48N_F)^{\frac{1}{6}}(\hbar/M\omega)^{\frac{1}{2}}$[16]

$$(15N_B g/4\pi M\omega^2)^{\frac{1}{5}} = (48N_F)^{\frac{1}{6}}(\hbar/M\omega)^{\frac{1}{2}} \quad (3.26)$$

This gives;

$$g = \frac{21.1N_F^{\frac{5}{6}}}{N_B}.\hbar\omega a_0^3 \quad (3.27)$$

3.4 Parameters

Table 3.1 shows a list of parameters of the experiments with a^{87}Rb–^{40}K boson-fermion mixture.

Table 3.1: List of parameters of experiments with ^{87}Rb-^{40}K Bose-Fermi mixture (Rothel, S., 2006)

Parameters	Hamburg experiment [23]	Florence experiment [14,24,25]
mass of ^{87}Rb atom $m_B = 14.43 \times 10^{-26}$ kg		
mass of ^{40}K atom $m_F = 6.636 \times 10^{-26}$ kg		
s-wave scattering length (bosons ↔ bosons) $a_{BB} = (5.238 \times 10^{-9} \pm 0.002)$ m		
s-wave scattering length (bosons ↔ fermions) $a_{BF} = -15.0 \times 10^{-9}ma_{BF} = (-20.0 \times 10^{-9} \pm 0.8)$ m		
radial trap frequency (bosons)	$\omega_{B,r} = 2\pi. 257$ Hz	$\omega_{B,r} = 2\pi. 215$ Hz
axial trap frequency (bosons)	$\omega_{B,z} = 2\pi. 11.3$ Hz	$\omega_{B,z} = 2\pi. 16.3$ Hz
radial trap frequency (fermions)	$\omega_{F,r} = 2\pi.\ 379$ Hz	$\omega_{F,r} = 2\pi.\ 317$ Hz
axial trap frequency (fermions)	$\omega_{F,z} = 2\pi.\ 16.7$ Hz	$\omega_{F,z} = 2\pi.\ 24.0$ Hz
number of bosons	$N_B = 10^6$	$N_B = 2 \times 10^5$
number of fermions	$N_F = 7.5 \times 10^5$	$N_F = 3 \times 10^4$

The list of parameters which have were used in the calculations in this thesis given below. g, is chosen to give maximal overlap between the two atomic clouds

$N_F = 10^3$

$N_B = 10^7$

$M_B == 1.45 \times 10^{-25} Kg$

$M_F = 6.636 \times 10^{-26} Kg$

$$M = \frac{M_F M_B}{M_F + M_B} = 4.54559 \times 10^{-26} Kg$$

$\omega = 2\pi \times 216\text{Hz}$

$g = 0.00066724 \hbar \omega a_0^3$

$$\hbar = \frac{h}{2\pi} = 1.0545 \times 10^{-34} Js$$

$$a_0 = \left(\frac{h}{M\omega}\right)^{\frac{1}{2}}$$

$$h = \frac{g}{2}, h = g, h = \frac{3g}{2}$$

3.5 Calculations

The density distribution of fermions is given by the equation (3.24), such that,

$$n_F(r) = \frac{\left\{\frac{2M}{\hbar^2}\left[(6N_F)^{\frac{1}{3}}\hbar\omega - \left(1 - \frac{h}{g}\right)\frac{1}{2}M\omega^2 r^2 - \frac{h}{g}\left[\frac{15}{8\pi}N_B g\left(\frac{m\omega^2}{2}\right)^{3/2}\right]^{\frac{2}{5}}\right]\right\}^{\frac{3}{2}}}{6\pi^2} \quad (3.28)$$

$$\frac{2M}{\hbar^2} = \frac{2 \times 4.54559 \times 10^{-26} Kg}{(1.0545 \times 10^{-34})^2}$$

$$= \frac{9.09118 \times 10^{-26+34+34}}{1.1109}$$

$$= 8.17574 \times 10^{42} Kg/J^2 s^2$$

and,

$$(6N_F)^{\frac{1}{3}}\hbar\omega = (6 \times 10^3)^{\frac{1}{3}} \times 1.0545 \times 10^{-34}\mathrm{Js} \times 2\pi \times 216\mathrm{s}^{-1}$$

$$= 2.6 \times 10^{-30}\mathrm{J}$$

and,

$$\left(1 - \frac{h}{g}\right)\frac{1}{2}\mathrm{M}\omega^2\mathrm{r}^2 = \left(1 - \frac{h}{g}\right) \times \frac{1}{2} \times 4.54559 \times 10^{-26} \times 2\pi \times 216 \times 2\pi \times 216 \times \mathrm{r}^2$$

$$= \left(1 - \frac{h}{g}\right) \times 4.186272 \times 10^{-20} \times \mathrm{r}^2$$

and,

$$\frac{h}{g}\mu = \frac{h}{g}\left[\frac{15}{8\pi} N_B g \left(\frac{m\omega^2}{2}\right)^{3/2}\right]^{2/5}$$

$$= \frac{h}{g}\left[\frac{15}{8\pi} N_B g \left(\frac{m\omega^2}{2}\right)^{3/2}\right]^{2/5}$$

and,

$$\mu = 3.9572 \times 10^{-30} J/Kg$$

$$g = 0.00066724\hbar\omega\mathrm{a}_0^3$$

$$= 6.6724 \times 10^{-4} \times 1.0545 \times 10^{-34} \times 2\pi \times 216 \times \left(\frac{\hbar}{\mathrm{M}\omega}\right)^{\frac{3}{2}}$$

$$= 6.6724 \times 10^{-4} \times 1.0545 \times 10^{-34} \times 2\pi \times 216$$

$$\times \left(\frac{1.0545 \times 10^{-34}}{4.54559 \times 10^{-26} \times 2\pi \times 216}\right)^{\frac{3}{2}}$$

$$= 2.13397 \times 10^{-52} J^2 s/Kg$$

Hence, equation 3.28 becomes;

$n_F(r)$

$$= \frac{\left\{8.18 \times 10^{40}\left[2.6 \times 10^{-30}\mathrm{J} - \left(1 - \frac{h}{g}\right) \times 4.186 \times 10^{-20} \times \mathrm{r}^2 - \frac{h}{g} \times 3.957 \times 10^{-30}\right]\right\}^{\frac{3}{2}}}{6\pi^2}$$

$n_F(r)$

$$= \frac{\left\{8.18 \times 10^{40}\left[2.6 \times 10^{-30}\mathrm{J} - \left(1 - \frac{h}{g}\right) \times 4.186 \times 10^{-20} \times \mathrm{r}^2 - \frac{h}{g} \times 3.957 \times 10^{-30}\right]\right\}^{\frac{3}{2}}}{59.21}$$

For bosons,

$$n_B(r) = \frac{\mu - V_{ext}(r) - h.n_F(r)}{g} \tag{3.27}$$

$$n_B(r) = \frac{3.957 \times 10^{-30} - 4.186 \times 10^{-20} \times \mathrm{r}^2 - h.n_F(r)}{2.13397 \times 10^{-52}} \tag{3.28}$$

For $h < g$ $(h = \frac{1}{2}g)$

$n_F(r)$

$$= \frac{\left\{8.18 \times 10^{40}\left[2.6 \times 10^{-30}\mathrm{J} - \left(1 - \frac{\frac{1}{2}g}{g}\right) \times 4.186 \times 10^{-20} \times \mathrm{r}^2 - \frac{\frac{1}{2}g}{g} \times 3.957 \times 10^{-30}\right]\right\}^{\frac{3}{2}}}{59.21}$$

$n_F(r)$

$$= \frac{\left\{8.18 \times 10^{40}\left[2.6 \times 10^{-30}\mathrm{J} - \frac{1}{2} \times 4.186 \times 10^{-20} \times \mathrm{r}^2 - \frac{1}{2} \times 3.957 \times 10^{-30}\right]\right\}^{\frac{3}{2}}}{59.21}$$

$n_F(r)$

$$= \frac{\{8.18 \times 10^{40}[2.6 \times 10^{-30}\mathrm{J} - 2.096 \times 10^{-20} \times \mathrm{r}^2 - 1.9785 \times 10^{-30}]\}^{\frac{3}{2}}}{59.21} \tag{3.29}$$

$$n_B(r) = \frac{3.957 \times 10^{-30} - 4.186 \times 10^{-20} \times \mathrm{r}^2 - \frac{1}{2} g. n_F(r)}{2.13397 \times 10^{-52}}$$

$$n_B(r) = \frac{3.957 \times 10^{-30} - 4.186 \times 10^{-20} \times \mathrm{r}^2}{2.13397 \times 10^{-52}} - 0.5. n_F(r)$$

$$n_B(r) = -1.9616 \times 10^{32} - 0.5. n_F(r) \tag{3.30}$$

Table 3.2: The calculated number of bosons and fermions from the trap centre for $\boldsymbol{h} < g \ (h = \frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{g})$

Distance from the trap Centre (r)	$n_F(r) \times 10^{31}$	$n_B(r) \times 10^{31}$
-5	-14.9875	-12.222
-4	-7.6737	-15.7791
-3	-3.2373	-17.9973
-2	-0.9592	-19.1364
-1	-0.119	-19.6101
0	0	-19.616
1	-0.119	-19.6101
2	-0.9592	-19.1364
3	-3.2373	-17.9973
4	-7.6737	-15.7791
5	-14.9875	-12.222

For $h = g$

$n_F(r)$

$$= \frac{\left\{8.18 \times 10^{40} \left[2.6 \times 10^{-30}\mathrm{J} - \left(1 - \frac{g}{g}\right) \times 4.186 \times 10^{-20} \times \mathrm{r}^2 - \frac{g}{g} \times 3.957 \times 10^{-30}\right]\right\}^{\frac{3}{2}}}{59.21}$$

$$n_F(r) = \frac{\{8.18 \times 10^{40}[2.6 \times 10^{-30}\mathrm{J} - 0 \times 4.186 \times 10^{-20} \times \mathrm{r}^2 - 3.957 \times 10^{-30}]\}^{\frac{3}{2}}}{59.21}$$

$$n_F(r) = \frac{\{8.18 \times 10^{40}[2.6 \times 10^{-30}\mathrm{J} - 0 \times \mathrm{r}^2 - 1.9785 \times 10^{-30}]\}^{\frac{3}{2}}}{59.21} \quad (3.31)$$

$$n_B(r) = \frac{3.957 \times 10^{-30} - 4.186 \times 10^{-20} \times \mathrm{r}^2 -. n_F(r)}{2.13397 \times 10^{-52}}$$

$$n_B(r) = \frac{3.957 \times 10^{-30} - 4.186 \times 10^{-20} \times \mathrm{r}^2}{2.13397 \times 10^{-52}} - n_F(r)$$

$$n_B(r) = -1.9616 \times 10^{32} - n_F(r) \quad (3.32)$$

Table 3.3: The calculated number of bosons and fermions from the trap centre for $h = g$

Distance from the trap Centre (r)	$n_F(r) \times 10^{31}$	$n_B(r) \times 10^{31}$
-5	1.6565	-19.16
-4	1.6565	-19.16
-3	1.6565	-19.16
-2	1.6565	-19.16
-1	1.6565	-19.16
0	1.6565	-19.16
1	1.6565	-19.16
2	1.6565	-19.16
3	1.6565	-19.16
4	1.6565	-19.16
5	1.6565	-19.16

For $h < g$ $(h = \frac{3}{2} g)$

$$n_F(r) = \frac{\left\{8.18 \times 10^{40} \left[2.6 \times 10^{-30}\mathrm{J} - \left(1 - \frac{\frac{3}{2}g}{g}\right) \times 4.186 \times 10^{-20} \times \mathrm{r}^2 - \frac{\frac{3}{2}g}{g} \times 3.957 \times 10^{-30}\right]\right\}^{\frac{3}{2}}}{59.21}$$

$$n_F(r) = \frac{\left\{8.18 \times 10^{40} \left[2.6 \times 10^{-30}\mathrm{J} + \frac{1}{2} \times 4.186 \times 10^{-20} \times \mathrm{r}^2 - \frac{3}{2} \times 3.957 \times 10^{-30}\right]\right\}^{\frac{3}{2}}}{59.21}$$

$$n_F(r) = \frac{\{8.18 \times 10^{40}[2.6 \times 10^{-30}\mathrm{J} + 2.096 \times 10^{-20} \times \mathrm{r}^2 - 1.9785 \times 10^{-30}]\}^{\frac{3}{2}}}{59.21} \quad (3.33)$$

$$n_B(r) = \frac{3.957 \times 10^{-30} - 4.186 \times 10^{-20} \times \mathrm{r}^2 - \frac{3}{2} g.\, n_F(r)}{2.13397 \times 10^{-52}}$$

$$n_B(r) = \frac{3.957 \times 10^{-30} - 4.186 \times 10^{-20} \times \mathrm{r}^2}{2.13397 \times 10^{-52}} - 1.5.\, n_F(r)$$

$$n_B(r) = -1.9616 \times 10^{32} - 1.5.\, n_F(r) \quad (3.34)$$

Table 3.4: The calculated number of bosons and fermions from the trap centre for $\boldsymbol{h} > g \ (h = \frac{\mathbf{3}}{\mathbf{2}}\boldsymbol{g})$

Distance from the trap Centre (r)	$n_F(r) \times 10^{31}$	$n_B(r) \times 10^{31}$
-5	14.9875	-42.0973
-4	7.6737	-31.1266
-3	3.2373	-24.4719
-2	0.9592	-21.0548
-1	0.119	-19.7945
0	0	-19.616
1	0.119	-19.7945
2	0.9592	-21.0548
3	3.2373	-24.4719
4	7.6737	-31.1266
5	14.9875	-42.0973

CHAPTER FOUR:

RESULTS AND DISCUSSIONS

4.1 Introduction

In this chapter, the results obtained are reported and analysis is done in order to obtain the density profiles for bosonic and fermionic components of the mixture. A detailed discussion based on those results is also done in this chapter.

The Bose and Fermi components can separate under variation of the strength of the boson-boson and boson-fermion interaction. In the Thomas-Fermi approximation for both components, the density distributions can be obtained by solving the coupled equations

$$V_{ext}(r) + g.n_B(r) + h.n_F(r) = \mu \tag{4.1}$$

$$\frac{\hbar^2}{2M}\left(6\pi^2 n_F(r)\right)^{2/3} + V_{ext}(r) + h.n_B(r) = E_F \tag{4.2}$$

The solution of equations 4.1 and 4.2 is obtained by insertion of the density distribution of bosons in equation 4.1 in equation 4.2 and numerically searching for energies μ and E_F yielding the desired number of particles. If fermions are very few, we can neglect the third term in the equation 4.1 and combine the resulting equation with equation 4.2 to get the equation of fermions as,

$$(6N_F)^{\frac{1}{3}}\hbar\omega = \frac{\hbar^2\left(6\pi^2 n_F(r)\right)^{2/3}}{2M} + \left[1 - \frac{h}{g}\right]V_{ext}(r) + \frac{h}{g}\mu \tag{4.3}$$

The strength of the boson-boson interaction was chosen to give the maximal overlap between the two clouds. This was done by equating Thomas Fermi expression for radius of Bose Condensate and the radius of zero temperature Fermi gas.

Modification of the interactions between the atoms especially its scattering length, using Feshbach resonance (Chin, C., et. al., 2010) and variation of external fields (Fedichev, P. O., et. al., 1996, Inouye, S., *et al*, 1998) has been reported. This has allowed the tuning of scattering length through positive and negative values. Positive values only were used in calculations in this research. The Thomas-Fermi expression for the Fermi energy of N_F fermions in a harmonic potential is $E_F = (6N_F)^{\frac{1}{3}}\hbar\omega$ (Butts, D. A., and Rokhsar, D. S., 1997).

The density distributions of harmonically trapped Bose Condensate and Fermi gas are obtained using experimental parameters of MIT experiments (Mølmer, K., 1998) given in Table3.1 and parameters used in this thesis on equation 4.3. In these experiments a Bose condensate of about 10^7 atoms may enclose roughly 10^3 fermions and vice versa.

4.2 Density distribution for bosons and fermions for $h < g$ $(h = \frac{1}{2}g)$

Figure 4.1(a) shows the density distribution of fermions as a function of the distance from the trap centre for $h < g$. From the figure, it can be seen that the fermionic particles has the largest number at the centre of the radial trap, i.e. at r=0. Their density falls when the distance from the trap centre is increased.

Figure 4.1(b) shows the density distribution of bosons as a function of the distance from the trap centre for $h < g$. From the figure it can be seen that the bosonic particles has the least number at the centre of the radial trap. Their density increases when the distance from the trap centre is increased.

Since they are trapped in a radial trap, then for $h < g$, the fermions will occupy the central part of the trap and will form something like a ball at the centre. The number of bosons will be very minimal at the centre but increases towards the periphery of the trap. It will look as if the bosons are enveloping the fermions. Therefore the fermions will be a shell immersed inside the bose condensate.

The trapping of bosonic and fermionic atoms is via Feshbach resonance. This is done by subjecting the atoms to an external magnetic field. The effective magnetic confinement experienced by atoms depends on the derivative of magnetic field (Ferlaino, 2004). The atoms will experience different cylindrical harmonic potential since they are trapped using different frequencies.

When $h < g$, the fermions experience a potential minimum at the center of the trap and therefore they populate around the center of the trap (r=0). Their number reduces as one moves away from the center. This is shown in the figure 4.1 (a). Bosons on the other end are expelled from the center of the trap. Their numbers increase as the distance from the trap center is increased. This is shown in the Figure 4.1(b). Since the particles were trapped in a radial trap and the number of fermions was small enough, they constituted a 'core' entire enclosed within the Bose condensate. The oscillation in the fermion density distribution near the trap center reflects the matter wave modulation in the outermost shell. In the center of the trap, the fermions in the lowest energy state experience a vanishing potential for h=0. The bosons are expelled from the trap center, minimizing their interaction energy by spreading in a shell around the fermionic bubble. The fermionic component is compressed, having a higher peak and density covering a smaller portion of the trapping volume. A similar behavior has been noted for bi-condensate systems (Miyakawa, T., et. al., 2000 and Griffin, A., 1996). The two quantum gases are said to be truly interpenetrating. In this case a Bose condensate of about 10^{36} atoms may enclose roughly 10^3 Fermions. For $h < g$, the distribution is as shown in the Fig 4.1

(a) (b)

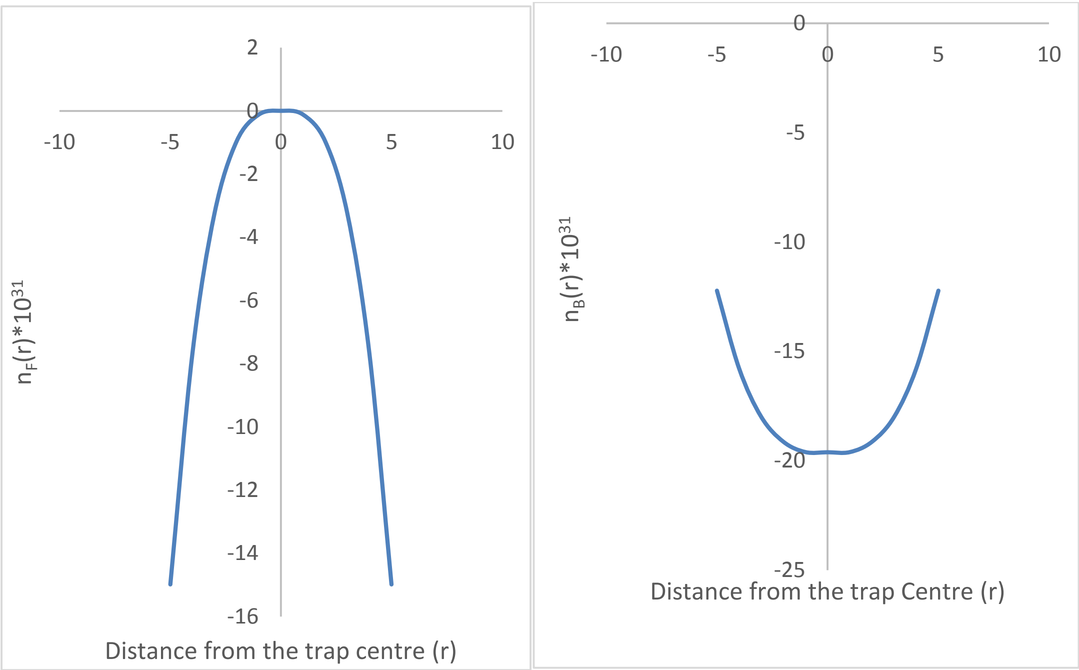

Figure 4.1; (a) shows the density distribution for fermions and (b) bosons at zero kelvin temperature for $h < g$ $(h = \frac{1}{2}g)$; Source (Author, 2016)

4.3 Density distribution for bosons and fermions for $h = g$

Figure 4.2(a) shows the density distribution of fermions as a function of the distance from the trap centre for $h = g$. From the figure it can be seen that the fermionic particles has a constant value for all the values of the distance from the trap centre. It implies that at any given point in the radial trap, we have equal number of fermions.

Figure 4.2(b) shows the density distribution of bosons as a function of the distance from the trap centre for $h = g$. From the figure it can be seen that the bosonic particles are uniformly distributed in the trap.

Thomas Fermi theory provides of a functional form for kinetic energy of non-interacting electron gas. One of the essential features predicted in the Thomas Fermi Approximation is the existence of a plateau of a constant fermionic density throughout the distribution. This phenomenon also appears in this quantum treatment. When $h = g$, the fermions have a constant density throughout the Bose condensate , as shown in the figure 4.2.

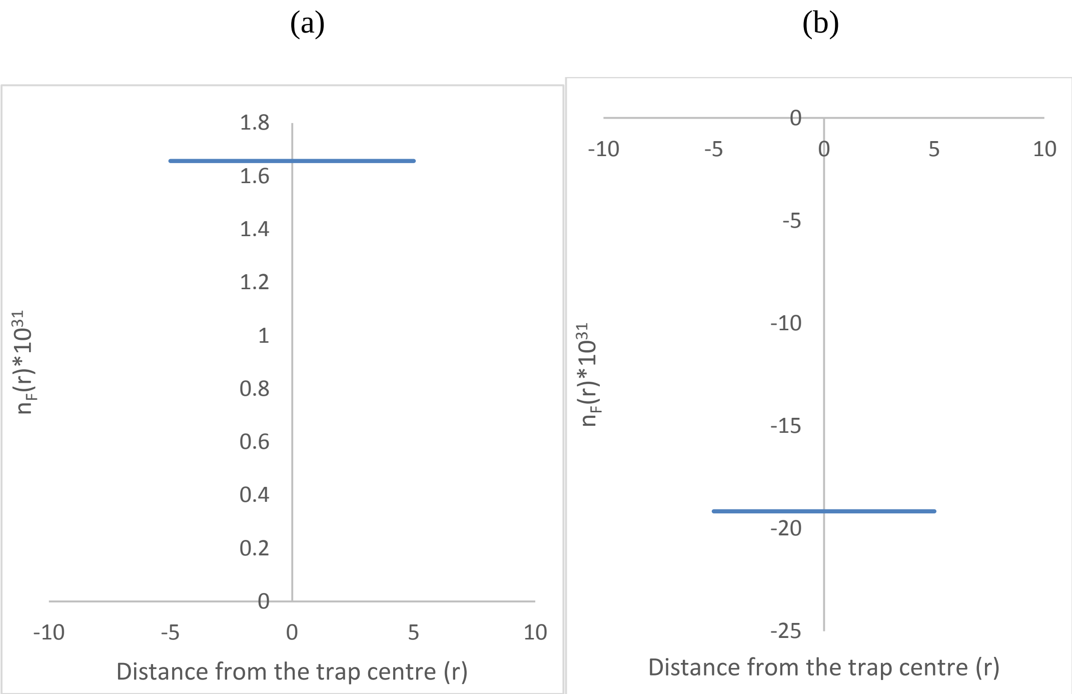

Figure 4.2; (a) shows the density distribution for fermions and (b) bosons at zero kelvin temperature for $h = g$; Source; (Author, 2016)

4.4 Density distribution for bosons and fermions for $h > g$ ($h = \frac{3}{2}g$)

Figure 4.3(a) shows the density distribution of fermions as a function of the distance from the trap centre for $h > g$. From the figure it can be seen that the number of fermionic particles increases as the distance from the trap centre is increased.

Figure 4.3(b) shows the density distribution of bosons as a function of the distance from the trap centre for $h > g$. From the figure it can be seen that the number of bosonic particles are concentrated at the centre of the trap.

When $h > g$, the effective potential for the fermions is that of an in invented harmonic oscillator having a minimum at the edge of the Bose condensate, where the fermions localize as a “shell” wrapped round the condensate. If the outer part is composed of fermions, and the number of fermions exceeds the limit of the Bose Condensate, then the atoms in the inner core will experience a stronger confining potential. The distribution is as shown in the figure 4.3. The fermions have been expelled from the Centre of the trap Centre and their number increase as the distance from the trap is increased.

We note that the semi-classical description gives a qualitatively correct description, in that it reliably predicts the phase separation.

We notice that as the bosons are expelled from the center of the trap, forming a ’mantle’ around the fermions, the fermionic component is compressed, having a higher peak density and covering a smaller portion of the trapping volume. A similar behavior has been noted for bi-condensate systems (Miyakawa, T., et. al., 2000 and Griffin, A., 1996). One of the essential features predicted in the Thomas-Fermi approximation is the existence of a ’plateau’ of constant fermion density through the boson distribution for $h = g$ As illustrated by Fig. 4.3.

(a) (b)

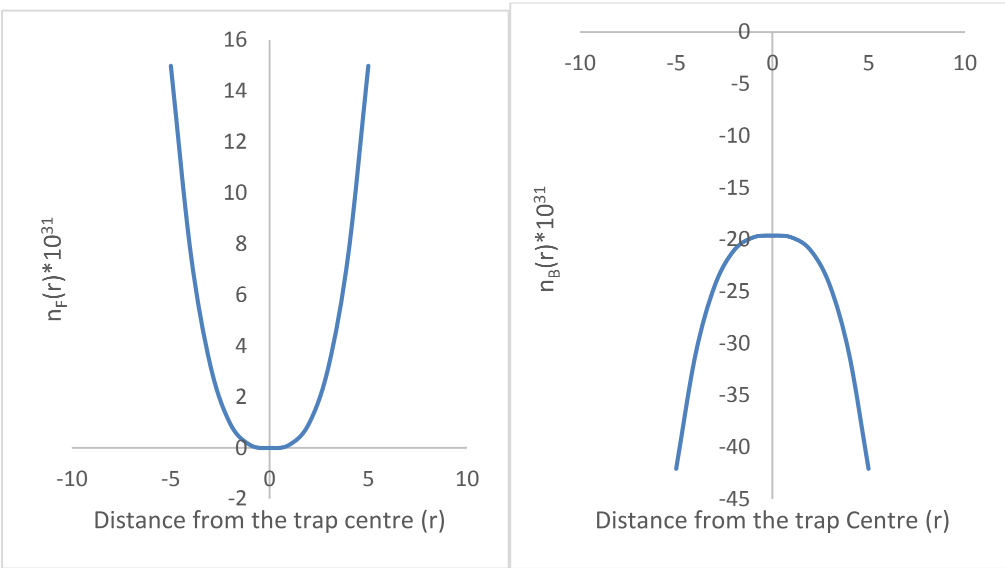

Figure 4.3; (a) shows the density distribution for bosons and (b) fermions at zero kelvin temperature for $h > g$ $(h = \frac{3}{2}g)$; Source; (Author, 2016)

This phenomenon also appear in our quantum mechanical treatment, although with the parameters chosen it does not involve quite as many particles as obtained from the semi-classical calculations in (Mølmer, K., 1998). It is interesting to compare the above mentioned results with those obtained by treating the fermions in the Thomas-Fermi approximation. We note that the semi-classical description gives a qualitatively correct description, in that it reliably predicts the phase separation.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this thesis, zero temperature ground state of a mixture of boson and fermion gases in isotropic trapping potentials have been investigated and results presented. Bosonic atoms possess according to the spin-statistics theorem integer Spin. Fermions have half integer spin, so in general there are at least two Fermionic components – one with spin up the other one with spin down – for spin one–half Fermions. Due to the Spin-alignment in magnetic traps the spin degree of freedom is frozen for the Fermions and we could treat the Fermions as one–component particles. In experiments with cold atomic gases, the dilute limit provides a very good approximation and thus the particle–particle interactions are dominated by *s*–wave scattering. Fermion–Fermion *s*–wave scattering is prohibited in this setting.

With these assumptions, within the framework of density functional theory the ground state energy could be incorporated to derive the density field equations beyond a mean-field level for systems of Bose–Fermi mixtures in a trap. More precisely, we have determined the density profiles in the Boson–Fermion scattering length. We solved these equations numerically for experimentally relevant parameters for ^{40} K–^{87}Rb mixtures and discussed the results.

According to the possible different combinations of intraspecies and interspecies attractive interactions, the system displays a rich phase structure and a simultaneous transition to demixing in the Boson–Fermion sector. The issue of component separation has been addressed by solving coupled equations for the spatial density of the two species. The calculations have confirmed the existence of three distinct states of the system under variation of the ratio of the interaction strengths h/g. For small values of this parameter, the gases are interpenetrable, overlapping throughout the occupied volume of the trap, as their mutual repulsion is not strong enough to cause separation. When the coupling strength h exceeds the strength of the boson-boson interaction, the fermions are expelled from the center of the trap. This is in agreement with the previous work (Nygaard, N. and Molmer, K., 1999). The spatial configuration in this case depends on the symmetry of the trapping potential. In an isotropic trap, the separated phase is rotationally symmetric; the fermions constitute a spherical shell wrapped around a centrally compressed Bose condensate. In the limiting case $h = g$, there exists the possibility for the fermions to have a constant spatial density where the bosons are localized.

5.2 Recommendations

An aspect of this work is the availability of an almost isolated degenerate Fermi gas through the complete separation of the two species. The trapped, degenerate Fermi gas is interesting in view of the possibility of a BCS transition when two spin states are trapped simultaneously (Bruun, G. M., *et. al*., 1998, Stoof, H. T. C., *et. al*., 1996 and Albus, A.P., *et.al*., 2003) and because of the analogies between this system and both atomic nuclei and the interior of neutron stars.

The details of sympathetic cooling of the Fermi gas to the degeneracy level through thermal contact with the Bose condensate are of great importance in further research

(Timmermans, E. and Cˆot´e, R., 1998). In general the investigation of the cooling ability of the condensate should not be restricted to fermionic impurities. In view of the recent trapping of simple molecules in both optical (Hirofumi, S., *et al*, 1998) and magnetic (Weistein, J. D., *et al*, 1998) potentials, also more complex solutes with several internal degrees of freedom pose an interesting challenge for future research.

Another direction worth noticing is the prospect of trapping a boson-fermion mixture in the periodic potential of an optical lattice (Berg-Sørensen, K. and Molmer, K., 1998) both in its own right and as a study of solid state phenomena. With quantum gases well beyond the degeneracy level a complete filling of the potential wells may well be expected (Mackie, M., *et. al*., 2004).

It should be mentioned that in this work, systems with a positive coupling strength h have been considered, allowing the interaction between the species to become attractive. This is known to induce a dramatic change in the macroscopic behavior of the system as it becomes unstable against collapse for large negative values of h (Albus, A.P., *et. al*., 2003).This phenomenon can be investigated in detail using the numerical procedure developed in this work.

Besides these fundamental theoretical aspects related to the theory of quantum phase transitions, Bose–Fermi mixtures in optical lattices are also promising candidates to observe BCS phase transitions and qualify for potential applications in the physics of quantum information. Bose–Fermi mixtures could in fact allow for new possibilities of quantum information processing in optical lattices. The Fermions would be suitable for storage of quantum information due to their non–interacting behavior, whereas the Bosons could be used to let the systems interact and perform operations.

The complexity of the systems considered in this thesis can be extended in various ways. For example one can consider unpolarized spin–1/2 fermions. Spin unpolarized systems are becoming of more experimental relevance, because one can nowadays also trap by purely optical means. In this case the Fermions are not necessarily spin– aligned. Calculations are very similar to the present situation with the main difference that one has to include the effects of the direct interactions of fermions with different spins. This would correspond to having a third scattering length a_{FF}.

As already pointed out, quasi one or two dimensional systems are expected to have very special properties. The methods used in this thesis could potentially also be used to those systems if applied with some care. An interesting perspective would be to investigate how the system behaves not only in the presence of Cooper pairing but also of ordinary molecule formation. Very recently there were even some speculations about the possibility of an atom–molecule Cooper pairing in Bose–Fermi mixtures (Mackie, M., *et. al.*, 2004). Some aspects of quasi-chemical equilibrium theory on molecular formation in atomic gas mixtures may be used to study the problem.

In the calculations presented in this thesis, two-body interactions have been used via the scattering length between boson-boson and boson-fermions interactions. However, it may be appropriate to use three-body interactions, although incorporating three-body will very much complicate the theory. One could consider three-body repulsive interactions among a pair of bosons, or three-body interaction among a pair of bosons and a fermion, or three-body interaction among a pair of fermions and a boson. In such a case the energy contribution to the Hamiltonian can be written as $\frac{\epsilon}{6}\sum_i n_i\,(n_i - 1)(n_i - 2)$, where $n_i$is the particle number operator for bosons at any site i , $i$is the three-body repulsive interaction strength between the bosons . Similar terms can be written for interaction between a pair of bosons and a fermion, and a pair of fermions and a boson. Many-body perturbation theory can be used to calculate the energy of the assembly (Bo-Cun, C. and Yunbo, Z., 2008, Mahan, G.D., 2000 and Fetter, A. L. and Walecka, J. D., 2003)

Depending on the relative magnitude of the types of interactions considered, one can obtain conditions for the kind of phase transitions and the phases (like super-fluid phase) that can exist in the mixture of bosons and fermions. Such studies will involve more complicated and advanced many-body theory. This study will be taken up in my advanced PhD thesis. It will be important to consider the effect of attractive boson-fermion interactions, and that of the repulsive boson-fermion interactions on the density profiles of bosons, fermions and boson-fermion (Robert, R., 2014) mixture.

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