International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 5, Issue 9, 2017, PP 34-38
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
DOI: http://dx.doi.org/10.20431/2347-3142.0509003
www.arcjournals.org

On Norm of Elementary Operator of Length Two

Denis Njue King'ang'i

Department of Mathematics and Computer Science, University of Eldoret, Eldoret, Kenya

***Corresponding Author:** *Denis Njue King'ang'I, Department of Mathematics and Computer Science, University of Eldoret, Eldoret, Kenya*

Abstract: *Norms of elementary operators have been studied by many mathematicians. Varied results have been gotten using different approaches. In this paper we use maximal numerical range to express the norm of an elementary operator of length two in terms of its coefficient operators.*

Keywords: *Elementary operator, Maximal numerical range, Jordan elementary operator.*

1. Introduction

Let H be a complex Hilbert space and $B(H)$ be the set of bounded linear operators on H. An *elementary operator*, $E_n: B(H) \longrightarrow B(H)$ is defined as,

$$E_n(W) = \sum_{i=1}^{n} T_i W S_i,$$

for all $W \in B(H), T_i, S_i$ being fixed elements of $B(H)$. When $n = 1$, then we have a *basic elementary operator*, $E_n: B(H) \longrightarrow B(H)$, defined as $E_1(W) = TWS$, for all $W \in B(H)$ and $T,\ S$ fixed in $B(H)$. The basic elementary operator is usually denoted by $M_{T,S}$. When $n = 2$, we obtain the *elementary operator of length two*, defined as,

$$E_2(W) = T_1 W S_1 + T_2 W S_2,$$

for all $W \in B(H)$ and T_i, S_i fixed in $B(H)$ for $i = 1,2$. This paper has determined the norm of the elementary operator of length two. The *Jordan elementary operator*, $U_{T,S}: B(H) \longrightarrow B(H)$, defined as $U_{T,S}(W) = TWS + SWT$ for all $W \in B(H)$ and $T,\ S$ fixed in $B(H)$, is an example of elementary operators.

The *maximal numerical range* of $T \in B(H)$ is the set,

$$W_0(T) = \{\lambda \in \mathbb{C}: \langle Tx_n, x_n\rangle \longrightarrow \lambda, \|x_n\| = 1, \|Tx_n\| \longrightarrow \|T\|\},$$

while the *maximal numerical range* $W_S(T^*S)$ of T^*S relative to S is defined as,

$$W_S(T^*S) = \left\{\lambda \in \mathbb{C}: \exists\{x_n\} \subseteq H, \|x_n\| = 1, \lim_{n\to\infty}\langle T^*Sx_n, x_n\rangle = \lambda, \lim_{n\to\infty}\|Sx_n\| = \|S\|\right\},$$

where T^* is the Hilbert adjoint of T.

For any $x, y \in H$, the rank one operator, $x \otimes y \in B(H)$, is defined by $(x \otimes y)(z) = \langle z, y\rangle x$, for all $z \in H$.

This paper employs the concept of the maximal numerical range to determine the lower bound of the norm of elementary operator E_2, and also to determine the conditions under which the norm of this operator is expressible in terms of the norms of its coefficient operators in $B(H)$. The approach used by Barraa and Boumazgour [1] is employed in obtaining our results.

2. The Norm of the Jordan Elementary Operator

Let H be a complex Hilbert space, $B(H)$ be the algebra of bounded linear operators on H, and $T, S \in B(H)$ be fixed. Recall that the Jordan elementary operator, $U_{T,S}: B(H) \longrightarrow B(H)$, is defined as $U_{T,S}(W) = TWS + SWT$, for all $W \in B(H)$.

Several people have attempted to determine the norm of this operator. Mathieu [4], in 1990 proved that in the case of prime C*-algebras, the lower bound of the norm of $U_{T,S}$ can be estimated by $\|U_{T,S}\| \geq \frac{2}{3}\|T\|\|S\|$. In 1994, Cabrera and Rodriguez [2], proved that $\|U_{T,S}\| \geq \frac{1}{20412}\|T\|\|S\|$, for prime JB*-algebras.

On their part, Stacho and Zalar [5], in 1996, worked on the standard operator algebra (which is a sub-algebra of $B(H)$ that contains all finite rank operators). They first showed that the operator $U_{T,S}$ actually represents a Jordan triple structure of a C*-algebra. They also showed that if A is a standard operator algebra acting on a Hilbert space H, and $T, S \in A$, then $\|U_{T,S}\| \geq 2(\sqrt{2}-1)\|T\|\|S\|$. They later (1998), proved that $\|U_{T,S}\| \geq \|T\|\|S\|$ for the algebra of symmetric operators acting on a Hilbert space. They attached a family of Hilbert spaces to standard operator algebra and used the inner products in them to obtain their results.

Barraa and Boumazguor [1], in the year 2001 used the concept of the numerical range of T relative to S, denoted by $W_S(T^*S)$, to obtain their results. They employed the idea of finite rank operators to show that if $T, S \in B(H)$ with $S \neq 0$, then

$$\|U_{T,S}\| \geq \sup_{\lambda \in W_S(T^*S)} \left\{ \left\| \|S\|T + \frac{\bar{\lambda}}{\|S\|} S \right\| \right\}.$$

As a consequent of this, they proved that if $0 \in W_S(T^*S) \cup W_T(S^*T)$, then $\|U_{T,S}\| \geq \|T\|\|S\|$. They also showed that if $\|T\|\|S\| \in W_T(S^*T) \cap \in W_{T^*}(ST^*)$, then $\|U_{T,S}\| = 2\|T\|\|S\|$.

3. Norm of Elementary Operator of Length Two

Kingangi et al [3] in 2014 used finite rank operators to determine the norm of the elementary operator E_2. They showed that for an operator $W \in B(H)$ with $\|W\| = 1$ and $W(x) = x$ for all unit vectors $x \in H$;

$$\|E_2\| = \sum_{i=1}^{2} \|T_i\|\|S_i\|.$$

Below, we present more results on the norm of this operator by employing the concept of the maximal numerical range. In the first two results, we determine the lower bound of the norm of E_2 while in the last result, the norm of E_2 is expressed in terms of the norms of its coefficient operators.

Theorem 3.1. *Let E_2 be an elementary operator of length two on $B(H)$. Then,*

$$\|E_2\| \geq \sup_{\lambda \in W_{S_1}(S_2{}^*S_1)} \left\{ \left\| \|S_1\|T_1 + \frac{\bar{\lambda}}{\|S_1\|} T_2 \right\| \right\},$$

where S_i, T_i are fixed elements of $B(H)$ for $i = 1,2$.

Proof. Let $\{x_n\}_{n\geq 1}$ be a sequence of unit vectors in a Hilbert space H and $y \otimes x_n \in B(H)$ be a rank-one operator on H, y be a unit vector in H, defined by $(y \otimes x_n)(x) = \langle x, x_n \rangle y$ for all $x \in H$. The maximal numerical range of $S_2^* S_1$ relative to S_1 is given as

$$W_{S_1}(S_2^*S_1) = \left\{ \lambda \in \mathbb{C} : \exists \{x_n\} \subseteq H, \|x_n\| = 1, \lim_{n\to\infty} \langle (S_2^*S_1)x_n, x_n \rangle = \lambda, \lim_{n\to\infty} \|S_1 x_n\| = \|S_1\| \right\},$$

where $S_1, S_2 \in B(H)$.

Now,

$$\|(E_2(y\otimes S_1x_n))x_n\| = \|(\sum_{i=1}^{2} M_{T_i,S_i}(y\otimes S_1x_n))x_n\|$$
$$\leq \|\sum_{i=1}^{2} M_{T_i,S_i}(y\otimes S_1x_n)\|\|x_n\|$$
$$\leq \|\sum_{i=1}^{2} M_{T_i,S_i}\|\|y\otimes S_1x_n\|$$
$$\leq \|\sum_{i=1}^{2} M_{T_i,S_i}\|\|y\|\|S_1\|\|x_n\|$$
$$= \|\sum_{i=1}^{2} M_{T_i,S_i}\|\|S_1\|$$

Therefore,

$$\|\sum_{i=1}^{2} M_{T_i,S_i}\|\|S_1\| \geq \|(\sum_{i=1}^{2} M_{T_i,S_i}(y\otimes S_1x_n))x_n\|$$
$$= \|(T_1(y\otimes S_1x_n)S_1 + T_2(y\otimes S_1x_n)S_2)x_n\|$$
$$= \|T_1(y\otimes S_1x_n)S_1x_n + T_2(y\otimes S_1x_n)S_2x_n\|$$
$$= \|\langle S_1x_n, S_1x_n\rangle T_1y + \langle S_2x_n, S_1x_n\rangle T_2y\|$$
$$= \|\|S_1x_n\|^2T_1y + \langle x_n, (S_2^*S_1)x_n\rangle T_2y\|$$

Thus;

(3.1) $$\|\sum_{i=1}^{2} M_{T_i,S_i}\| \geq \frac{1}{\|S_1\|}\|\|S_1x_n\|^2T_1y + \langle x_n, (S_2^*S_1)x_n\rangle T_2y\|.$$

Taking limits as $n \longrightarrow \infty$ we obtain,

$$\lim_{n\longrightarrow\infty} \langle x_n, (S_2^*S_1)x_n\rangle = \overline{\lim_{n\longrightarrow\infty} \langle (S_2^*S_1)x_n, x_n\rangle} = \overline{\lambda},\ \lim_{n\longrightarrow\infty} \|S_1x_n\| = \|S_1\|.$$

Therefore,

$$\|\sum_{i=1}^{2} M_{T_i,S_i}\| \geq \left\|\|S_1\|T_1y + \frac{\overline{\lambda}}{\|S_1\|}T_2y\right\|,$$

and this is true for any $\lambda \in W_{S_1}(S_2^*S_1)$ and for any unit vector $y \in H$.

Now, consider the set;

$$\left\{\left\|\|S_1\|T_1y + \frac{\overline{\lambda}}{\|S_1\|}T_2y\right\| : \lambda \in W_{S_1}(S_2^*S_1), y \in H, \|y\| = 1\right\},$$

where T_1, T_2, S_1, S_2 are fixed elements of $B(H)$.

Then we have $\|\sum_{i=1}^{2} M_{T_i,S_i}\| \geq sup_\lambda \left\{\left\|\|S_1\|T_1y + \frac{\overline{\lambda}}{\|S_1\|}T_2y\right\| : \lambda \in W_{S_1}(S_2^*S_1), y \in H, \|y\| = 1\right\}$.

But;

$sup_\lambda \left\{\left\|\|S_1\|T_1y + \frac{\overline{\lambda}}{\|S_1\|}T_2y\right\| : \lambda \in W_{S_1}(S_2^*S_1), y \in H, \|y\| = 1\right\} = sup_\lambda \left\{\left\|\|S_1\|T_1 + \frac{\overline{\lambda}}{\|S_1\|}T_2\right\| : \lambda \in\right.$ *Ws1S2*S1.*

Therefore, $\|\sum_{i=1}^{2} M_{T_i,S_i}\| \geq sup_\lambda \left\{\left\|\|S_1\|T_1 + \frac{\overline{\lambda}}{\|S_1\|}T_2\right\| : \lambda \in W_{S_1}(S_2^*S_1)\right\}$. That is,

$$\|E_2\| \geq sup_\lambda \left\{ \left\| \|S_1\|T_1 + \frac{\bar{\lambda}}{\|S_1\|} T_2 \right\| : \lambda \in W_{S_1}(S_2^* S_1) \right\},$$

and this completes the proof.

Corollary 3.2. *Let H be a complex Hilbert space and T_i, S_i be bounded linear operators on H for $i = 1,2$. Let $0 \in W_{S_1}(S_2^* S_1) \cup W_{S_2}(S_1^* S_2)$. Then,$\|E_2\| \geq \|T_1\|\|S_1\|$, where E_2 is as defined earlier.*

Proof. Let $0 \in W_{S_1}(S_2^* S_1) \cup W_{S_2}(S_1^* S_2)$. Then $0 \in W_{S_1}(S_2^* S_1)$ or $0 \in W_{S_2}(S_1^* S_2)$, and therefore, either there is a sequence $\{x_n\}_{n>1}$ of unit vectors in H such that $\lim_{n\to\infty}\langle (S_2^* S_1)x_n, x_n\rangle = 0$ and $\lim_{n\to\infty}\|S_1 x_n\| = \|S_1\|$, or, there is a sequence $\{y_n\}_{n>1}$ of unit vectors in H such that $\lim_{n\to\infty}\langle (S_1^* S_2)y_n, y_n\rangle = 0$ and $\lim_{n\to\infty}\|S_2 y_n\| = \|S_2\|$. Recall that in inequality 3.1, we obtained;

$$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \geq \frac{1}{\|S_1\|} \| \|S_1 x_n\|^2 T_1 y + \langle x_n, (S_2^* S_1)x_n\rangle T_2 y\|.$$

This is equivalent to;

(3.2) $$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \geq \frac{1}{\|S_1\|} \| \|S_1 y_n\|^2 T_1 y + \langle (S_1^* S_2)y_n, y_n\rangle T_2 y\|,$$

considering the sequence $\{y_n\}_{n>1}$.
Taking limits in either inequality (3.1) or inequality (3.2), we obtain

$$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \geq \left\| \|S_1\|T_1 y \right\|,$$

and this is true for all unit vectors $y \in H$.
Now, consider the set $\left\{ \left\| \|S_1\|T_1 y \right\| : y \in H, \|y\| = 1 \right\}$, where T_1 and S_1 are bounded linear operators on H. Then we have;

$$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \geq sup\left\{ \left\| \|S_1\|T_1 y \right\| : y \in H, \|y\| = 1 \right\}.$$

But $sup\left\{ \left\| \|S_1\|T_1 y \right\| : y \in H, \|y\| = 1 \right\} = \|S_1\|\|T_1\|$. Therefore, $\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \geq \|S_1\|\|T_1\|$, or $\|E_2\| \geq \|T_1\|\|S_1\|$, and this completes the proof.

In the next theorem, the conditions under which the norm of the elementary operator E_2 is expressible in terms of the norms of the corresponding coefficient operators on H is given.

Theorem 3.3. *Let H be a complex Hilbert space and T_i, S_i be bounded linear operators on H for $i = 1,2$. Let E_2 be an elementary operator of length two. If $\|T_1\|\|T_2\| \in W_{T_1^*}(T_2 T_1^*)$ and $\|S_1\|\|S_2\| \in W_{S_2}(S_1^* S_2)$, then, $\|E_2\| = \sum_{i=1}^{2}\|T_i\|\|S_i\|$.*

*Proof.*Suppose that $\|T_1\|\|T_2\| \in W_{T_1^*}(T_2 T_1^*)$ and $\|S_1\|\|S_2\| \in W_{S_2}(S_1^* S_2)$. Then there are two sequences $\{x_n\}_{n>1}$ and $\{y_n\}_{n>1}$ of unit vectors in H such that $\lim_{n\to\infty}\langle (T_2 T_1^*)x_n, x_n\rangle = \|T_1\|\|T_2\|$, $\lim_{n\to\infty}\|T_1^* x_n\| = \|T_1\|$, and $\lim_{n\to\infty}\langle (S_1^* S_2)y_n, y_n\rangle = \|S_1\|\|S_2\|$, $\lim_{n\to\infty}\|S_2 y_n\| = \|S_2\|$. Since $|\langle (T_2 T_1^*)x_n, x_n\rangle| \leq \|T_1^* x_n\|\|T_2^* x_n\|$ and $|\langle (S_1^* S_2)y_n, y_n\rangle| \leq \|S_2 y_n\|\|S_1 y_n\|$, then $\lim_{n\to\infty}\|T_2^* x_1\| = T2$ and lim*n→∞S1yn=S1.*
For each $n \geq 1$, we have;

$$\left\|\left(E_2(x_n \otimes y_n)\right)T_1^* y_n\right\|^2 = \|(S_1(x_n \otimes y_n)T_1 + S_2(x_n \otimes y_n)T_2)T_1^* y_n\|^2$$
$$= \|S_1(x_n \otimes y_n)T_1 T_1^* y_n + S_2(x_n \otimes y_n)T_2 T_1^* y_n\|^2$$
$$= \|\langle T_1 T_1^* y_n, y_n\rangle S_1 x_n + \langle T_2 T_1^* y_n, y_n\rangle S_2 x_n\|^2$$
$$= \|\langle T_1^* y_n, T_1^* y_n\rangle S_1 x_n + \langle T_2 T_1^* y_n, y_n\rangle S_2 x_n\|^2$$
$$= \| \|T_1^* y_n\|^2 S_1 x_n + \langle T_2 T_1^* y_n, y_n\rangle S_2 x_n\|^2$$
$$= \langle \|T_1^* y_n\|^2 S_1 x_n + \langle T_2 T_1^* y_n, y_n\rangle S_2 x_n, \|T_1^* y_n\|^2 S_1 x_n + \langle T_2 T_1^* y_n, y_n\rangle S_2 x_n\rangle$$
$$= \| \|T_1^* y_n\|^2 S_1 x_n\|^2 + 2Re\langle \langle T_2 T_1^* y_n, y_n\rangle S_2 x_n, \|T_1^* y_n\|^2 S_1 x_n\rangle + |\langle T_2 T_1^* y_n, y_n\rangle|^2 \|S_2 x_n\|^2$$
$$= \|T_1^* y_n\|^4 \|S_1 x_n\|^2 + 2\|T_1^* y_n\|^2 Re\langle T_2 T_1^* y_n, y_n\rangle\langle S_2 x_n, S_1 x_n\rangle + |\langle T_2 T_1^* y_n, y_n\rangle|^2 \|S_2 x_n\|^2$$
$$= \|T_1^* y_n\|^4 \|S_1 x_n\|^2 + 2\|T_1^* y_n\|^2 Re\langle T_2 T_1^* y_n, y_n\rangle\langle S_1^* S_2 x_n, x_n\rangle + |\langle T_2 T_1^* y_n, y_n\rangle|^2 \|S_2 x_n\|^2.$$

Now, $\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\|\|T_1\| \geq \left\|\left(\sum_{i=1}^{2} M_{T_i,S_i}(x_n \otimes y_n)\right) T_1^* x_n\right\|$. Therefore,

$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\|^2 \|T_1\|^2 \geq$
$\|T_1^* y_n\|^4 \|S_1 x_n\|^2 + 2\|T_1^* y_n\|^2 Re\langle T_2 T_1^* y_n, y_n\rangle\langle S_1^* S_2 x_n, x_n\rangle + |\langle T_2 T_1^* y_n, y_n\rangle|^2 \|S_2 x_n\|^2.$
Letting $n \longrightarrow \infty$, we obtain,

$$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\|^2 \|T_1\|^2 \geq \|T_1\|^4 \|S_1\|^2 + 2\|T_1\|^2 \|T_2\| \|T_1\| \|S_1\| \|S_2\| + \|T_2\|^2 \|T_1\|^2 \|S_2\|^2.$$

That is,

$$\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\|^2 \geq \|T_1\|^2 \|S_1\|^2 + 2\|T_2\| \|T_1\| \|S_1\| \|S_2\| + \|T_2\|^2 \|S_2\|^2.$$

But $\|T_1\|^2 \|S_1\|^2 + 2\|T_2\| \|T_1\| \|S_1\| \|S_2\| + \|T_2\|^2 \|S_2\|^2 = (\|T_1\| \|S_1\| + \|T_2\| \|S_2\|)^2$.
Thus, $\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \geq \|T_1\| \|S_1\| + \|T_2\| \|S_2\| = \sum_{i=1}^{2} \|T_i\| \|S_i\|$.
That is, $\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \geq \sum_{i=1}^{2} \|T_i\| \|S_i\|$.
Clearly, $\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| \leq \sum_{i=1}^{2} \|T_i\| \|S_i\|$, and thus, $\left\|\sum_{i=1}^{2} M_{T_i,S_i}\right\| = \sum_{i=1}^{2} \|T_i\| \|S_i\|$.
That is, $\|E_2\| = \sum_{i=1}^{2} \|T_i\| \|S_i\|$.

4. Conclusion

Much remain to be done in determining the norm of E_2. One may also attempt this problem working on tensor product

References

[1] M. Baraa and M. Baumazgour, *a lower bound of the norm of the operator $x \longmapsto axb + bxa$*, Extracta mathematicae 16 (2001), 223-227.

[2] M. Cabrera and A. Rodriguez, *Non-degenerate ultraprime jordan banach algebras: a zelmanorian treatment*. Proc.london.math.soc 69 (1994), 576-604.

[3] D. N. King'ang'i, J. O. Agure and F. O. Nyamwala, *On the norm of elementary operator,* Advances in Pure Mathematics 4 (2014), 309-316.

[4] M. Mathieu, *More properties of the product of two derivations of a c*-algebra*, Bull.austral.math.soc 42(1990), 115-120.

[5] L. L. Stacho and B. Zalar, *On the norm of the Jordan elementary operator in standard operator algebra*, Publ.math.debreen 49 (1996), 127-134.

Author's Biography

Dr. Denis Njue King'ang'i holds a PhD in pure mathematics. He is currently lecturing at the University of Eldoret, Kenya. His research interests are in Operator Theory and Graph Theory.

Citation: *D. N. King'ang'i, " On Norm of Elementary Operator of Length Two ", International Journal of Scientific and Innovative Mathematical Research, vol. 5, no. 9, p. 34-38, 2017., http://dx.doi.org/10.20431/ 2347-3142.0509003*