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ON THE NORM OF AN ELEMENTARY OPERATOR OF FINITE LENGTH IN A C^* ALGEBRA

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Abstract: Properties of elementary operators have been studied over the past years especially the norm aspect. Various results have been obtained on elementary operators of different lengths using different approaches. In this paper, we determine the norm of an elementary operator of length n in a C*algebra using finite rank operators.We will review known results on Jordan and general elementary operators which are useful in getting our result.

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Key words: *C*algebra ;Elementary Operator; Finite rank Operator.*

1. Introduction

Elementary operators have been studied in the past few years and their norms obtained by different researchers. Bounds of these norms have been put into consideration in many papers among other aspects. While finding the upper bound of the norm of this operator is trivial by the triangular inequality, determining the lower bound is challenging. This has led to the lower bound of the norm of elementary operator attracting the attention of many researchers in operator theory.

A C^* algebra $\mathcal{A}$is a Banach algebra over the field of complex numbers,in which is defined an involution $*: \mathcal{A} \rightarrow \mathcal{A}$, such that,$\|xx^*\| = \|x\|^2$for all xin thegiven B^*algebra. The property $\|xx^*\| = \|x\|^2$is reffered to as the C^* identity. Onewrites$x^*$for the image of an element x of $\mathcal{A}$.

For a C^* algebra $\mathcal{A}$, an operator $T : \mathcal{A} \rightarrow \mathcal{A}$ is called an *elementary operator* if T can be expressed in the form; $Tx = \Sigma A_i x B_i$ with A_i and $B_i (1 \leq i \leq l)$ in $\mathcal{A}$ and $l \in \mathbb{N}$. For $\mathcal{A}$ a C^*algebra, one may allow A_i and B_i to be in the multiplier algebra of A.

If $A, B \in \mathcal{A}$, we define a *basic elementary operator;*

$$M_{A,B}: \mathcal{A} \to \mathcal{A} \; by \; M_{A,B}(x) = AxB$$

Let H be a Hilbert space, $\mathcal{L}(H)$algebra of all bounded linear operators on H and

$M_{A,B}: \mathcal{L}(H) \to \mathcal{L}(H)$be the basic elementary operator then the operator;

$$U_{A,B} = M_{A,B} + M_{B,A}$$

is called the *Jordan elementary operator*.

A finite-rank operator$T : U \to V$ is a bounded operator between Banach spaces such that its range is finite dimensional. In the Hilbert space case, it can be written in the form ;

$T\,x = \sum n_i = 1\ \alpha_i \langle x, v_i \rangle u_i$for all $x \in U$, where by$u_i \in V$, and $v_i \in U'$ are bounded linear functionals on the space U.

In this paper we determine the norm of an elementary operator of length$n$ in aC^*algebra using finite rank operators.

2. Norm of the Jordan Elementary Operator

Norms of elementary operators were studied by Mathieu[5] in 1990who worked on prime C^*algebras.They proved the following;

Theorem 2.1[Mathieu,1990]

Let $\mathcal{A}$ be a prime C^*algebra and$A, A', B, B', B'', C \in \mathcal{A}$.Then;

$2\|A\|\|B\|\|C\| \le \|M_{A,B} + M_{B',A'}\|\|C\| + \|M_{B,C} + M_{A',B''}\|\|A\| + \|M_{A,B''} + M_{B',C}\|\|A'\|$. In particular they proved that;

$$\|M_{A,B} + M_{B,A}\| \ge {}^{2}/_{3}\,\|A\|\|B\|$$

Cabrera and Rodriguez[4] in 1994 worked on a Jordan elementary operator. Stacho and Zalar[6] in 1996 gave the lower bound for standard operator algebras by attaching a family of Hilbert spaces to a standard operator and using inner product on them in order to obtain a supremum type estimate. They showed that if $\mathcal{A}$ is a standard operator algebra acting on a Hilbert space $H, A, B \in \mathcal{A}$ and$A = B$, then

$$\|U_{A,B}\| \ge 2(\sqrt{2} - 1)\|A\|\|B\|.$$

Barraa and Boumazgour [2] in 2001 gave the lower bound of a two sided multiplication operator using the concept of the numerical range as shown in the following three results;

Theorem 2.2

Let $A, B \in \mathcal{L}(H)$ with $B \ne 0.$ Then

$$\|M_{A,B} + M_{B,A}\| \ge \sup_{\lambda \in W_B(A^*B)} \left\| \|B\|A + \frac{\bar{\lambda}}{\|B\|} B \right\|$$

Corollary 2.3

If $0 \in W_B(A^*B) \cup W_A(B^*A)$ then;

$$\|M_{A,B} + M_{B,A}\| \ge \|A\|\|B\|$$

Proposition 2.4

If$\|A\|\|B\| \in W_A(B^*A) \cap W_{A^*}(BA^*)$, then;

$$\|M_{A,B} + M_{B,A}\| = \|M_{A,B}\| + \|M_{B,A}\| = 2\|A\|\|B\|$$

3. Norm of General Elementary Operator

Timoney[1], in 2007 used numerical ranges and the tracial geometric mean to obtain an approximation of E_n. In 2008 Nyamwala and Agure[2] used the spectral resolution theorem to calculate the norm of E_n induced by normal operators in a finite dimensional Hilbert Space. In 2014 King'ang'iet al [4] gave the norm of a general elementary operator of length two using the finite rank operators.Their work forms the basis of the results in this paper.

The following is their result;

Theorem 3.1

Let H be a complex Hilbert space and $\mathcal{L}(H)$ the algebra of all bounded linear operators on H. Let E_2 be the elementary operator on $\mathcal{L}(H)$defined above. If for an operator$W \in \mathcal{L}(H)$with $\|W\| = 1$, we have $W(x) = x$for all unit vectors x$\in H$, then

$\|E_2\| = \sum_{i=1}^{2} \|A_i\|\|B_i\|$.

In 2017 King'ang'i[7] gave the results below for an elementary operator of length two.

Theorem 3.2

Let E_2 be an elementary operator of length two on $\mathcal{L}(H)$ then,

$\|E_2\| \geq \sup_{\lambda \in W_{S_1}(S_2{}^{*}S_1)} \left\{ \left\| \|S_1\|T_1 + \frac{\bar{\lambda}}{\|S_1\|} T_2 \right\| \right\}$,where$S_i, T_i$ are fixed elements of $\mathcal{L}(H)$ for

$i = 1,2$, where$W_{S_1}(S_2{}^{*}S_1)$ is the maximal numerical range of $S_2{}^{*}S_1$ relative to S_1.

Corollary 3.3

Let H be a complex Hilbert space and T_i, S_i bounded linear operators on H for

$i = 1,2$.Let 0$\in W_{S_1}(S_2{}^{*}S_1) \cup W_{S_2}(S_1{}^{*}S_2)$. Then $\|E_2\| \geq \|T_1\|\|S_1\|$, where E_2 is as defined earlier.

Theorem 3.4

Let H be a complex Hilbert space and T_i, S_i bounded linear operators on H for

$i = 1,2$. Let E_2 be an elementary operator of length two.

If $\|T_1\|\|T_2\| \in W_{T_1{}^{*}}(T_2 T_1{}^{*})$ and $\|S_1\|\|S_2\| \in W_{S_2}(S_1{}^{*}S_2)$ then, $\|E_2\| = \sum_{i=1}^{2} \|T_i\|\|S_i\|$

Theorem 3.5

Let E_2 be an elementary operator on $\mathcal{L}(H)$and $S_i, T_i \in L(H)$for $i = 1,2$. . If

$\|S_i\| \in W_o(S_i)$and$\|T_i\| \in W_o(T_i)$ for $1,2,3, \ldots . n$, then $\|E_2\| = \sum_{i=1}^{2} \|T_i\|\|S_i\|$.

In the next theorem we have our main result.We employ finite rank operators to obtain the norm of an elementary operator of length n.

Theorem 3.7

Let H be a complex Hilbert space and $\mathcal{L}(H)$ be the algebra of all bounded linear operators on H. Let E_n be the elementary operator on $\mathcal{L}(H)$. If for an operator $X \in \mathcal{L}(H)$ with $\|X\| = 1$, we have $X(f) = f$ for all unit vectors $f \in H$ then;

$$\|E_n\| = \sum_{i=1}^{n} \|A_i\|\|B_i\|, n \in \mathbb{N}$$

Proof:

E_n is defined as $E_n(X) = A_1XB_1 + A_2XB_2 + A_3XB_3 + \cdots + A_nXB_n$ for all

$X \in \mathcal{L}(H)$ and

$A_i, B_i \in \mathcal{L}(H), i = 1,2,3, \dots n$. We have;

$$\|E_n|\mathcal{L}(H)\| = sup\{\|E_n(X)\|: X \in L(H), \|X\| = 1\}$$

Therefore $\|E_n|\mathcal{L}(H)\| \geq \|E_n(X)\| \ \forall X \in \mathcal{L}(H), \|X\| = 1$.

Now, for all real $\varepsilon > 0$, we have $\|E_n|\mathcal{L}(H)\| - \varepsilon < \|E_n(X)\| \ \forall X \in \mathcal{L}(H), \|X\| = 1$.

Thus $\|E_n|\mathcal{L}(H)\| - \varepsilon < \|A_1XB_1\| + \|A_2XB_2\| + \|A_3XB_3\| + \cdots + \|A_nXB_n\|$

$$= \sum_{i=1}^{n} \|A_iXB_i\| \leq \sum_{i=1}^{n} \|A_i\|\|B_i\|$$

Letting $\varepsilon \to 0$ we have;

$\|E_n|\mathcal{L}(H)\| \leq \sum_{i=1}^{n} \|A_i\|\|B_i\|$..(1)

Next we show the reverse inequality, that is;

$\|E_n|\mathcal{L}(H)\| \geq \sum_{i=1}^{n} \|A_i\|\|B_i\|$.

Since $\|E_n(X)\| = sup\{\|E_n(X)f\|: f \in H, \|f\| = 1\}$ then we have

$\|E_n(X)\| \geq E_n(X)f \ \ \forall f \in H \ \|f\| = 1$.

Recall

$$E_n(X)f = (A_1XB_1 + A_2XB_2 + A_3XB_3 + \cdots + A_nXB_n)f$$

Let $u_i, v_i: H \to \mathbb{R}^+$ be functionals for $i = 1,2,3, \dots n$. Choose unit vectors $y, z \in H$ and let

$A_i = u_i \otimes y$ and $B_i = v_i \otimes y$ be finite rank operators on H for $i = 1,2,3, \dots n$ defined by

$A_if = (u_i \otimes y)f = u_i(f)y$ and $B_if = (v_i \otimes z)f = v_i(f)z$ for $f \in H$ with $\|f\| = 1$ for

$$i = 1,2,3, \dots n.$$

$$\| A_i\| = sup\{\|(u_i \otimes y)f\|: f \in H, \|f\| = 1\}$$

$$= sup\{\|u_i(f)y\|: f \in H, \|f\| = 1\}$$

$$= sup\{|u_i(f)|\|y\|: f \in H, \|f\| = 1\}$$

$$= sup\{|u_i(f)|: f \in H, \|f\| = 1\} = |u_i(f)|$$

That is, $\|A_i\| = |u_i(f)|$ for $= 1,2,3, \dots n$ where $f \in H, \|f\| = 1$.

Similarly, $\|B_i\| = |v_i(f)|$ for $= 1,2,3, \dots n$ where $f \in H, \|f\| = 1$.

Thus for all $f \in H, \|f\| = 1$, we have;

$$E_n(X)f = (A_1XB_1 + A_2XB_2 + A_3XB_3 + \cdots + A_nXB_n)f$$

$$= (A_1XB_1)f + (A_2XB_2)f + (A_3XB_3)f + \cdots + (A_nXB_n)f$$

$$= (u_1 \otimes y)X(v_1 \otimes z)f + (u_2 \otimes y)X(v_2 \otimes z)f + (u_3 \otimes y)X(v_3 \otimes z)f + \cdots + (u_n \otimes y)X(v_n \otimes z)f$$

$$= v_1(f)(u_1 \otimes y)X(z) + v_2(f)(u_2 \otimes y)X(z) + v_3(f)(u_3 \otimes y)X(z) + \cdots + v_n(f)(u_n \otimes y)X(z)$$

$$= v_1(f)u_1\big(X(z)\big)y + v_2(f)u_2\big(X(z)\big)y + v_3(f)u_3\big(X(z)\big)y + \cdots + v_n(f)u_n\big(X(z)\big)y$$

Recall $\|E_n|\mathcal{L}(H)\| = sup\{\|E_n(X)\|: for\ all\ X \in \mathcal{L}(H), \|X\| = 1\}$.

Hence, $\|E_n|\mathcal{L}(H)\| \geq \|E_n(X)\|: X \in \mathcal{L}(H), \|X\| = 1$.

Therefore,

$$\|E_n|\mathcal{L}(H)\|^2 \geq \left\|v_1(f)u_1\big(X(z)\big)y + v_2(f)u_2\big(X(z)\big)y + v_3(f)u_3\big(X(z)\big)y + \cdots + v_n(f)u_n\big(X(z)\big)y\right\|^2$$

$$= \langle v_1(f)u_1\big(X(z)\big)y + v_2(f)u_2\big(X(z)\big)y + v_3(f)u_3\big(X(z)\big)y + \cdots + v_n(f)u_n\big(X(z)\big)y, v_1(f)u_1\big(X(z)\big)y + v_2(f)u_2\big(X(z)\big)y + v_3(f)u_3\big(X(z)\big)y + \cdots + v_n(f)u_n\big(X(z)\big)y\rangle$$

$$\begin{aligned}= &\langle v_1(f)u_1\big(X(z)\big)y + v_2(f)u_2\big(X(z)\big)y + v_3(f)u_3\big(X(z)\big)y + \cdots + v_n(f)u_n\big(X(z)\big)y, v_1(f)u_1\big(X(z)\big)y\rangle \\ &+ \langle v_1(f)u_1\big(X(z)\big)y + v_2(f)u_2\big(X(z)\big)y + v_3(f)u_3\big(X(z)\big)y + \cdots \\ &+ v_n(f)u_n\big(X(z)\big)y, v_2(f)u_2\big(X(z)\big)y\rangle \\ &+ \langle v_1(f)u_1\big(X(z)\big)y + v_2(f)u_2\big(X(z)\big)y + v_3(f)u_3\big(X(z)\big)y + \cdots \\ &+ v_n(f)u_n\big(X(z)\big)y, v_3(f)u_3\big(X(z)\big)y\rangle + \cdots \\ &+ \langle v_1(f)u_1\big(X(z)\big)y + v_2(f)u_2\big(X(z)\big)y + v_3(f)u_3\big(X(z)\big)y + \cdots \\ &+ v_n(f)u_n\big(X(z)\big)y, v_n(f)u_n\big(X(z)\big)y\rangle\end{aligned}$$

$=\langle v_1(f)u_1\big(X(z)\big)y, v_1(f)u_1\big(X(z)\big)y\rangle + \langle v_2(f)u_2\big(X(z)\big)y, v_1(f)u_1\big(X(z)\big)y\rangle$

$+\langle v_3(f)u_3\big(X(z)\big)y, v_1(f)u_1\big(X(z)\big)y\rangle + \dots$

$+\langle v_n(f)u_n\big(X(z)\big)y, v_1(f)u_1\big(X(z)\big)y\rangle + \langle v_1(f)u_1\big(X(z)\big)y, v_2(f)u_2\big(X(z)\big)y\rangle$

$+\langle v_2(f)u_2\big(X(z)\big)y, v_2(f)u_2\big(X(z)\big)y\rangle + \langle v_3(f)u_3\big(X(z)\big)y, v_2(f)u_2\big(X(z)\big)y\rangle$

$+\dots + \langle v_n(f)u_n\big(X(z)\big)y, v_2(f)u_2\big(X(z)\big)y\rangle$

$+\langle v_1(f)u_1(X(z))y, v_3(f)u_3(X(z))y\rangle + \langle v_2(f)u_2(X(z))y, v_3(f)u_3(X(z))y\rangle$

$+\langle v_3(f)u_3(X(z))y, v_3(f)u_3(X(z))y\rangle + \cdots + \langle v_n(f)u_n(X(z))y, v_3(f)u_3(X(z))y\rangle + \cdots$

$+\langle v_1(f)u_1(X(z))y, v_n(f)u_n(X(z))y\rangle +$
$\langle v_2(f)u_2(X(z))y, v_n(f)u_n(X(z))y\rangle + \langle v_3(f)u_3(X(z))y, v_n(f)u_n(X(z))y\rangle +$
$\cdots + \langle v_n(f)u_n(X(z))y, v_n(f)u_n(X(z))y\rangle$

$=\|v_1(f)u_1(X(z))y\|^2 + v_2(f)u_2(X(z))v_1(f)u_1(X(z))\langle y,y\rangle + v_3(f)u_3(X(z))v_1(f)u_1(X(z))\langle y,y\rangle + \cdots +$

$v_n(f)u_n(X(z))v_1(f)u_1(X(z))\langle y,y\rangle + v_1(f)u_1(X(z))v_2(f)u_2(X(z))\langle y,y\rangle +$

$\|v_2(f)u_2(X(z))y\|^2 + v_3(f)u_3(X(z))v_2(f)u_2(X(z))\langle y,y\rangle + \cdots +$
$v_n(f)u_n(X(z))v_2(f)u_2(X(z))\langle y,y\rangle + v_1(f)u_1(X(z))v_3(f)u_3(X(z))\langle y,y\rangle +$
$v_2(f)u_2(X(z))v_3(f)u_3(X(z))\langle y,y\rangle + \|v_3(f)u_3(X(z))y\|^2 + \cdots + v_n(f)u_n(X(z))v_3(f)u_3(X(z))\langle y,y\rangle + \cdots +$
$v_1(f)u_1(X(z))v_n(f)u_n(X(z))\langle y,y\rangle + v_2(f)u_2(X(z))v_n(f)u_n(X(z))\langle y,y\rangle +$
$v_3(f)u_3(X(z))v_n(f)u_n(X(z))\langle y,y\rangle + \cdots + \|v_n(f)u_n(X(z))y\|^2$

$=$
$|v_1(f)|^2|u_1(X(z))|^2 + v_2(f)u_2(X(z))v_1(f)u_1(X(z)) + v_3(f)u_3(X(z))v_1(f)u_1(X(z)) + \cdots +$
$v_n(f)u_n(X(z))v_1(f)u_1(X(z)) + v_1(f)u_1(X(z))v_2(f)u_2(X(z)) + |v_2(f)|^2|u_2(X(z))|^2 +$
$v_3(f)u_3(X(z))v_2(f)u_2(X(z)) + \ldots + v_n(f)u_n(X(z))v_2(f)u_2(X(z)) + v_1(f)u_1(X(z))v_3(f)u_3(X(z)) +$
$v_2(f)u_2(X(z))v_3(f)u_3(X(z)) + |v_3(f)|^2|u_3(X(z))|^2 + \cdots + v_n(f)u_n(X(z))v_3(f)u_3(X(z)) + \cdots +$
$v_1(f)u_1(X(z))v_n(f)u_n(X(z)) + v_2(f)u_2(X(z))v_n(f)u_n(X(z)) + v_3(f)u_3(X(z))v_n(f)u_n(X(z)) + \cdots +$
$|v_n(f)|^2|u_n(X(z))|^2$

$$= \{|v_1(f)||u_1(X(z))|\}^2 + \{|v_2(f)||u_2(X(z))|\}^2 + \{|v_3(f)||u_3(X(z))|\}^2 + \cdots + \{|v_n(f)||u_n(X(z))|\}^2 + v_1(f)u_1(X(z))v_2(f)u_2(X(z))$$

$+v_1(f)u_1(X(z))v_3(f)u_3(X(z)) + \cdots + v_{n-1}(f)u_{n-1}(X(z))v_n(f)u_n(X(z))$

Now ,since $v_1(f), v_2(f), \ldots v_n(f), u_1(X(z)), u_2(X(z)), \ldots, u_n(X(z))$ are all positive real numbers,we have ;

$v_1(f) = |v_1(f)| = \|B_1\|,\ v_2(f) = |v_2(f)| = \|B_2\|, \ldots, v_n(f) = |v_n(f)| = \|B_n\|$ and

$u_1(X(z)) = |u_1(X(z))| = \|A_1\|, u_2(X(z)) = |u_2(X(z))| = \|A_2\|, \ldots,$

$u_n(X(z)) = |u_n(X(z))| = \|A_n\|.$

Thus

$\|E_n|\mathcal{L}(H)\|^2 \geq$
$\{\|A_1\|\|B_1\|\}^2 + \{\|A_2\|\|B_2\|\}^2 + \{\|A_3\|\|B_3\|\}^2 + \cdots + \{\|A_n\|\|B_n\|\}^2 +$
$\|A_1\|\|B_1\|\|A_2\|\|B_2\| + \|A_1\|\|B_1\|\|A_3\|\|B_3\| + \cdots \|A_{n-1}\|\|B_{n-1}\|\|A_n\|\|B_n\| = \sum_{i=1}^{n}\{\|A_i\|\|B_i\|\}^2 +$
$\sum_{j=1}^{n}\sum_{k=1}^{n}\|A_j\|\|B_j\|\|A_k\|\|B_k\|$ for $j \neq k$.

Therefore, $\|E_n|\mathcal{L}(H)\|^2 \geq \{\|A_1\|\|B_1\| + \|A_2\|\|B_2\| + \|A_3\|\|B_3\| + \cdots + \|A_n\|\|B_n\|\}^2$

$$\|E_n|\mathcal{L}(H)\| \geq \|A_1\|\|B_1\| + \|A_2\|\|B_2\| + \|A_3\|\|B_3\| + \cdots + \|A_n\|\|B_n\| = \sum_{i=1}^{n}\|A_i\|\|B_i\|$$

Thus ,$\|E_n|\mathcal{L}(H)\| \geq \sum_{i=1}^{n}\|A_i\|\|B_i\|$…………………………………………………..…(2)

Hence combining 1 and 2 we have,

$$\|E_n|\mathcal{L}(H)\| = \sum_{i=1}^{n}\|A_i\|\|B_i\|$$

4. Conclusion

We have determined the norm of an elementary operator of an arbitrary lengthin a C* algebra using finite rank operators. Norm of an elementary operator of finite length may also be attempted using numerical ranges.

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