ON NUMERICAL RANGE OF SOME CLASSES OF OPERATORS IN HILBERT SPACES

By

Emmanuel Kipruto

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF SCIENCE OF PURE MATHEMATICS IN THE SCHOOL OF SCIENCE UNIVERSITY OF ELDORET, KENYA.

2021

DECLARATION

Declaration by the Student

This thesis is my original work and has not been submitted for any academic award in any institution; and shall not be reproduced in part or full, or in any format without prior written permission from the author and / or university.

Emmanuel Kipruto

SSCI/MAT/M/007/18 **Date**

Declaration by the Supervisors

This thesis has been submitted for examination with our approval as university supervisors.

Dr. Linety Muhati **Date**

Department of mathematics and computer science,

University of Eldoret, Kenya

Dr. Collins Meli. **Date**

Department of mathematics and computer science,

University of Eldoret, Kenya

DEDICATION

I dedicate this work to my wife Lucy Mukolwe and son Liam Kigen.

ABSRACT

The numerical range of operators in Hilbert spaces has been researched by several authors. The properties of the numerical range play an important role in identifying the behaviour of operators in Hilbert spaces. It is a well-known fact in operator theory that the spectrum of an operator is contained in the closure of the numerical range. The aim of our study was to establish the condition that gives the generalization that the spectrum of an operator is contained in the numerical range. Such spectral properties and the location of the numerical range in the complex plane were significant in determining the behavior of different classes of operators. We compared and analyzed known properties of the numerical range in the complex Hilbert space and narrowed this results to compact operators, spectroloid operators and partial isometries. Our results benefits other areas of mathematics and applied sciences such as quantum computing, physics and analysis. But more notably numerical range are used in engineering as rough estimates of eigenvalues of operators.

TABLE OF CONTENTS

LIST OF NOTATIONS

$\mathbb{K}$ – Field of complex or real numbers

$\mathbb{R}$- The field of real numbers.

$\mathbb{C}$-The field of complex numbers.

$\|.\|$-The norm function.

$\langle\cdot,\cdot\rangle$- Inner product function

$\mathcal{H}$- A complex Hilbert space.

$\mathcal{B}(\mathcal{H})$- Set of bounded linear operator acting on the Hilbert space.

K$(\mathcal{H})$- The set of compact operators.

T - An operator.

$\omega(T)$- Numerical radius of an operator T.

T^* - Adjoint of operator.

W(T)- Numerical range of an operator T.

σ(T)- The spectrum of an operator.

λ- Complex scalar.

r(T)- The spectral radius of T.

σ_c(T)- Continuous spectrum.

σ_R(T) - Residual spectrum.

σ_P (T) - Point spectrum.

$Con(S)$ - Convex hull of set S.

$\rho(T)$ - Resolvent set of T.

ACKNOWLEDGMENT

Am thankful to the almighty God for the good health he has provided me with throughout this research.

Am also very grateful to the overwhelming support accorded to me by my supervisors and lecturers; Dr. Linety Muhati, and Dr. Collins Melli.

I would also pass my gratitude to Prof J.M Khalagai for the guidance he accorded me throughout this project and also to Dr. D. King'ang'i for the important role he played throughout the lectures.

I will not fail to recognize my colleague S. Karani for the encouragements we accorded each other when doing this study.

Finally, I would pass my appreciation to the University of Eldoret for giving me the opportunity to undertake my postgraduate studies.

CHAPTER ONE

INTRODUCTION

1.1 Background

The numerical range of an operator T on a Hilbert space $\mathcal{H}$ is a set $W(\mathrm{T}) = \{\langle \mathrm{T}x, x\rangle : \|x\| = 1\}$, this set has been referred to by other researchers as the field of values or the hausdorff domain to range of values.

The first publication on the numerical range was done by Toeplitz in (1918), when he proved that the numerical range is convex set. But since then a lot of research has been done and the set of the numerical range is still undergoing rigorous research. Meng(1957) showed that if an operator is normal and its numerical range is closed then the extreme points of the numerical range are eigenvalues.

Shapiro J (2004) discussed the relation between eigenvalues and points on the boundary of the numerical range. He showed that 'corner points' of numerical range are eigenvalues and furthermore used the Hildebrandt theorem to show that eigenvalues on the boundary of the numerical range behaves just like the eigenvalues of normal operators.

1.2 Definitions and terminologies

Definition 1.2.1: A norm $\|.\|$ on a vector space V is a nonnegative real valued function with the following axioms holding for all $x,y \in$ V and $\lambda \in \mathbb{K}$.

$\|x\| \geq 0$

$\|x\|$ = 0 if and only if x = 0

$\|\lambda x\| = |\lambda|\|x\|$

$\|x + y\| \leq \|x\| + \|y\|$

The norm of an operator T is defined as $\|T\| = sup(\|Tx\|: x \in \mathcal{H}\ with\ \|x\| = 1)$

Definition 1.2.2: An inner product $\langle\cdot,\cdot\rangle$ on a vector space V is a function with the following axioms holding for all $x, y, z \in V$ and $\lambda \in \mathbb{K}$;

$\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and if $x = 0$

$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$

$\langle x, y\rangle = \langle\overline{y, x}\rangle$

Definition 1.2.3: An operator T is function acting on elements on the same space.

Definition 1.2.4: The operator T is said to be bounded if there exist some scalar $M > 0$ such that for all $x \in \mathcal{H}$ we have $\|Tx\| \leq M\|x\|$.

Definition 1.2.5: An operator T^* is called the adjoint of T if $\langle \mathrm{T}^* x, x\rangle = \langle x, Tx\rangle$ for all $x \in \mathcal{H}$.

Definition 1.2.6 Let $\mathcal{H}$ be a Hilbert space and T be an operator from $\mathcal{H}$ to $\mathcal{H}$. Then we have the following classes of operators.

(i) Self-adjoint operator (or hermitian) if $\mathrm{T}^* = \mathrm{T}$

(ii) Isometric if $\mathrm{T}^*\mathrm{T} = \mathrm{I}$

(iii) Normal operator if $\mathrm{T}^*\mathrm{T} = \mathrm{T}\mathrm{T}^*$

(iv) Unitary operator if $\mathrm{T}^*\mathrm{T} = \mathrm{T}\mathrm{T}^* = \mathrm{I}$.

(v) Partial isometries if $T\mathrm{T}^*\mathrm{T} = T$

(vi) Projection operator if $T^2 = T$

Definition 1.2.7: The numerical range of a bounded linear operator on a Hilbert space $\mathcal{H}$ is the set W (T) = $\{\langle \mathrm{T}x, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$. The numerical range is said to be closed when it contains the boundary points and its closure is denoted as $\overline{W(T)}$. There are some known properties of the numerical range i.e.

a) $W(T)$ is invariant under unitary similarity,

b) $W(T)$ lies in the closed disc of radius $\|T\|$ centered at the origin,

c) $W(T)$ contains all the eigenvalues of T

d) $W(T^*)$ = $(\bar{\lambda}$: $\lambda \epsilon$ W (T)

e) If α and β are complex numbers and T a bounded operator on $\mathcal{H}$ then,

W $(\alpha T + \beta I) = \alpha W$(T) $+ \beta$.

f) If $\mathcal{H}$ is finite dimensional then $W(T)$ is compact

Definition 1.2.8: The numerical radius is denoted as $w(T)$ and it is defined as $w(T) = sup\{|\lambda|: \lambda \in W(T)\}$.There are some known properties of the numerical radius i.e.

(a) $w(T) \geq 0$

(b) $w(\alpha T) = |\alpha|\, w(T)\, for\ all\ \alpha \in \mathbb{C}$.

(c) $w(T + S) \leq w(T) + w(S)\, for\ all\ T, S\ \mathcal{B}(\mathcal{H})$.

Definition 1.2.9: The spectrum of bounded linear operator T is defined as the set $\sigma(T) = \{\lambda \in \mathbb{C}: (\lambda I - \mathrm{T})^{-1}$ does not exist }.The spectrum has radius denoted as $\mathfrak{r}(T) = sup\{|\lambda|: \lambda \in \sigma(T)\ for\ all\ (\lambda I - T) \neq 0\}$.The compliment of the spectrum is called the resolvent set which is given by $\rho(T) = \{\mathbb{C} \setminus \sigma(T)\}$.The spectrum can be divided into the following three components

The point spectrum is denoted $\sigma_p(T) = \{\lambda \in \mathbb{C}: Tx = \lambda x\}$.

Continuous spectrum denoted has $\sigma_c(T) = \{\lambda \in \mathbb{C}$: $(\lambda$I $- T)^{-1}\}$ is bounded and $\overline{R(\lambda I - \mathrm{T})} = \mathcal{H}$.

Residual spectrum denoted as $\sigma_R(T) = \{\lambda \in \mathbb{C}: (\lambda I - T)^{-1}\}$ exists and range is not densely defined $\overline{R(\lambda I - \mathrm{T})} \neq \mathcal{H}$.

It has been shown that the $\sigma(T)$ can be used to identify some classes of operators, if we let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator then,

a) T is self adjoint iff $\sigma(T)$ is real

b) T is a projection iff the $\sigma(T) \in \{0,1\}$

c) T is unitary iff the set $\sigma(T) \in \{\mathbb{Z} \in \mathbb{C}: \|\mathbb{Z}\| = 1\}$

Definition 1.2.10. An operator T is defined as Normaloid if $\|\mathrm{T}\| = \mathrm{r}(\mathrm{T})$

Definition 1.2.11. An operator T is defined as Spectroloid if $\omega(\mathrm{T}) = r(\mathrm{T})$

Definition 1.2.12. Let X, Y be Hilbert spaces, a bounded linear operator

$T: X \to Y$ between Hilbert spaces is said to be compact if for every bounded sequence$\{x_n\}_{n\in\mathbb{N}} \subset X$, their exist subsequence of the sequence $\{Tx_n\}_{\ n\in\mathbb{N}}$ which converges in Y.

Definition 1.2.13: A set S is said to be convex if the line segment between any two points lies in S, that is if $x, y \in S$, then $\lambda x + (1-\lambda)$y ∈ S for all λ ∈[0,1]. Given any nonempty sets S there is the smallest convex set containing S denoted by Con(S) and is referred to as the convex hull of S. Equivalently, it is the intersection of all convex sets containing S.

Definition 1.2.14: a sequence x_1, x_2 ... ... of real numbers is called a Cauchy sequence if for every positive real numbers ε, there is a positive integer N such that for all natural numbers $m, n \geq N$ $|x_m - x_n| < \varepsilon$.

1.3 Statement of the problem

The research that has been done on the numerical range of operators in general has dealt with mainly topological properties such as convexity, closure and compactness. The role of the numerical range in identifying properties of operators and its location on the complex plane for specific classes of operators has not been well researched. In this thesis we investigated the properties of numerical range for spectroloid operators, compact operators and partial isometries

1.4 Objectives of the study

1.4.1 Main objective

To investigate the role of numerical range on properties of operators together with its location on the complex plane.

1.4.2 Specific objectives

The specific objectives of this research were as follows;

(i) To investigate numerical range properties on spectroloid operators.

(ii) To investigate numerical range properties on compact operators.

(iii) To investigate the invariance of numerical range under partial isometries.

1.5 Significance of the study

The numerical range has different applications to other areas of mathematics and applied sciences such as quantum computing, physics and analysis. But more notably numerical range are used in engineering as rough estimates of eigenvalues of operators, operators are also essential in formation of theories in quantum physics.

1.6 Research methodology

We compare the known properties of the numerical range for different operators and its location on the complex plane to be able to identify the operators using these properties. We narrowed our study to specific classes of operators i.e. Spectroloid, Compact and Partial isometries.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Different authors have significantly contributed to the study of numerical range of operators. Toeplitz (1918) did their research work on numerical range of operators and proved that the numerical range is convex. They further proved that the closure of the numerical range includes the spectrum and the convex hull of the spectrum. Generally more work has to be done in showing that the spectrum is contained in the numerical range of an operator. Toeplitz also showed that the numerical range has a numerical radius which is much bigger than the spectral radius of the operator since the closure of the numerical range contains the spectrum. J. P Williams (1967) did an extension to the closure of the numerical range and proved that, the spectrum of the product of a positive invertible operator and a selfadjoint operator is contained in the difference of the closure of their numerical ranges, similar investigation shows that the spectra product of an invertible compact operator with any other bounded linear operator is contained in the difference of the closure of the numerical range of the bounded operator and the numerical range of an invertible compact operator. J.P. Williams's (1969) in his paper of similarity and numerical range showed that the numerical range of positive operator and its spectrum lies on the real axis. M. Newman (1982) made his contribution on the numerical radius and spectral radius of unitary operators by asserting that they are invariant under unitary similarity

Karl Gustafson and K.M Rao (1995) proved that the numerical range of a self adjoint operator together with its closure lies on the real axis of the complex plane. Further they researched on products of the numerical radii of operators. Shapiro(2004) worked on numerical range of bounded linear operators on a two dimensional Hilbert space. He proved that the numerical range of a normal and non-normal operators will have different properties. Numerical range of non-normal operator brings out clearly an elliptic disc which agrees with Toeplitz theorem that numerical range is convex in shape. For the numerical range of normal operators he proved that it will either be a scalar multiple of the identity or a line segment joining the eigenvalue.

Takayuki Furuta (2001) gave some of the fundamental general properties of subclasses of partial isometries operators. Literature on the numerical range of this subclasses is scarce ranging from unitary operators, isometries to partial isometries. Wafula, A and J.M. Khalagai(2010), researched on the invariance of numerical range of operator under isometric transformation. They extended their research on the invariance of the numerical range under co-isometries. They also stated that if a partial isometry is either injective or has dense range then the numerical range is invariant under partial isometries.

According to the flow of literature review above we note that not many authors have directed their interest to specific classes of operators, this is the reason why the focus in this thesis was on spectroloids operators, compact operators and partial isometries.

CHAPTER THREE

THE NUMERICAL RANGE OF SPECTROLOID OPERATORS

3.1 Introduction

In this chapter we investigate properties of the numerical range on spectroloid operators. Spectroloids are operators in which the spectral radii equal the numerical radii and can be identified through certain subclasses of operators, starting with, Projections, Selfadjoint operators, normal operators, normaloid operators and spectroloid operators.

The following are the inclusions of the subclasses of spectroloid operators

{Projections }⊂{selfadjoint}⊂{normal}⊂{normaloid}⊂ $\{spectroloid\}$.

3.2 General properties of the numerical range of operators

The numerical range of any operator has the following known general properties that have been researched on. They include, closure, the spectral inclusion, and the relation between the numerical radius and the spectral radius as the following results show.

Proposition3.2.1. ($\boldsymbol{Karl\ Gustafson\ and\ K.M\ Rao, 1995}$). Let $T \in \mathcal{B}(\mathcal{H})$ be self adjoint operator and W (T) denote the numerical range.

Then W (T*) = $\{\bar{\lambda}:\ \lambda \epsilon$ W (T)$\}$

Proof:

For an operator $T \in \mathcal{B}(\mathcal{H})$ which is self-adjoint we have that;

$\langle Tx, x\rangle = \langle x, T^{*}x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$ for each $x \in \mathcal{H}$.Hence we have that complex number coincide with its complex conjugate. Hence W$(T^{*}) = \overline{W(T)}$ so is real thus $W(T) \subset \mathbb{R}$. ■

Lemma 3.2.2. $(\boldsymbol{Karl\ Gustafson\ and\ K.M\ Rao, 1995})$. Let T $\in \mathcal{B}(\mathcal{H})$ and $\sigma_p(T)$ denote point spectrum of T then, $\sigma_p(T) \subseteq W(T)$.

Proof:

Let $x \in \mathcal{H}$ be such that $\|x\| = \langle x, x\rangle = 1$, then it follows that,

$$W(T) = \langle Tx, x\rangle = \langle \lambda x, x\rangle \text{ for } \lambda \in \sigma_p(T)$$

We have $Tx = \lambda x$

$$\langle Tx, x\rangle = \langle \lambda x, x\rangle$$

$$= \lambda\langle x, x\rangle$$

$$= \lambda\|x\|^2$$

Thus $\lambda \in W(T)$. Hence $\sigma_p(T) \subseteq W(T)$

Similarly for $\bar{\lambda} \in \sigma_p(T^*)$ we have $Tx = \bar{\lambda}x$

$$\langle Tx, x\rangle = \langle \bar{\lambda}x, x\rangle$$

$$= \bar{\lambda}\langle x, x\rangle$$

$$= \bar{\lambda}\ \|x\|^2$$

Thus $\bar{\lambda} \in W(T)$. Hence $\sigma_P(T^*) \subseteq W(T)$.

Since both $\sigma_P(T^*)$ and $\sigma_p(T)$ are contained in the numerical range then $\bar{\lambda} = \lambda$ hence real. ■

Theorem 3.2.3.$(\boldsymbol{Toeplitz\ 1918})$. The numerical range closure encompasses the spectrum $i.e.\overline{W(T)} \supset \sigma(T)$

Proof:

Since $W(T)$ is convex in complex Hilbert space by $(Toeplitz)$.

Let $\mu \notin \overline{W(T)}$. Then for any unit vector x,

0< $d = d\big(\mu, \overline{W(T)}\big) \leq |\langle Tx, x\rangle - \mu| = |\langle T - \mu\rangle x, x| \leq \|\langle Tx - \mu x\rangle\|$,by Schwartz inequality.

So $\|Tx - \mu x\| \geq d\|x\|$ for unit vectors x. It follows that $\mu \in \rho(T)$ then $\bar{\mu} \in \sigma_p(T^*)$ that is there exist a unit vector x such that $T^*x = \bar{\mu}x$ so that $\bar{\mu} \in W(T^*)$. Equivalently, $\mu \in W(T)$ which contradicts to $\mu \notin \overline{W(T)}$. Hence, $\mu \in \sigma(T)$ and therefore $\sigma(T) \in \overline{W(T)}$. ■

Remark 3.2.4 We note that the following theorem by J.P Williams (1967) is an extension of theorem 3.2.3 above.

Theorem 3.2.5. ($\boldsymbol{J.P\ williams, 1967}$). Let T a positive operator and S self adjoint operator, on a Hilbert space $\mathcal{H}$. If $0 \notin W\overline{(T)}$ then $\sigma(T^{-1}S) \subset \overline{W(S)} - \overline{W(T)}$.

Proof:

Since $\sigma(T) \subset \overline{W(T)}$, then by hypothesis T^{-1} exists. Secondly, the identity

$$T^{-1}S - \lambda = T^{-1}(S - \lambda T)$$

Shows that if $\lambda \in \sigma(T^{-1}S)$, then $0 \in \sigma(S - \lambda T)$.Hence this implies that

$0 \in \overline{W(S - \lambda T)} \subset \overline{W(S)} - \lambda\overline{W(T)}$, and thus our desired result,

$\lambda \in \sigma(T^{-1}S) \subset \overline{W(S)} - \overline{W(T)}$

■

Remark 3.2.6. The numerical radius and the spectral radius helps in classifying different subclasses of spectroloid operators as it is shown below.

Proposition 3.2.7($\boldsymbol{Karl\ Gustafson\ and\ K.M\ Rao, 1995}$). For the spectral radius $\mathfrak{r}(T)$ of an operator T we have that $\mathfrak{r}(T) \leq \mathcal{w}(T)$

Proof:

Since $\sigma(T) \subset \overline{W(T)}$ and $\mathfrak{r}(T) = sup\{|\lambda|: \lambda \in \sigma(T)\ for\ all\ (\lambda I - T) \neq 0\}$ and $\mathcal{w}(T) = sup\{|\lambda|: \lambda \in W(T)\}$ then it follows that $\mathfrak{r}(T) \leq \mathcal{w}(T)$.

When the norm operator is equal to the spectral radii we have a class of operators called normaloid and in extension when the spectral radii equal the numerical radius we have a class of spectroloid operators. ■

3.3 Numerical range of the subclasses of spectroloid operators.

THEOREM3.3.1.$(\boldsymbol{Dada}, \boldsymbol{et\ al}, \mathbf{2016})$**.**Let $T \in \mathcal{B}(\mathcal{H})$ where

$$\mathbb{P} = \{P \in \mathcal{B}(\mathcal{H}: P\ is\ an\ orthogonl\ projection\ on\ \mathcal{H})\},$$

$\wp_{\mathcal{K}} = \{P \in \mathbb{P}: \dim\ (Ran(p) = \mathcal{K}\}, \mathcal{K} \in \mathbb{N}$

And $\wp_A = \{P \in \mathbb{P}: PA = AP, \dim\ (Ran(P)) < \infty\}$, then the following holds.

(i) If $\dim \mathcal{H} = \mathcal{K}$, then $W_{\wp_{\mathcal{K}}}(T) = \sigma(T)$.

(ii) If $\dim \mathcal{H} < \infty$, then $W_{\wp_{\mathcal{K}}}(T)$ is closed for $\mathcal{K} \geq 1$.

(iii) If $\dim \mathcal{H} > \mathcal{K}$,then $W_{\wp_{\mathcal{K}}}(T) = W(T)$.

Proof

(i) Is a direct consequence of $\wp_{\mathcal{K}} = \{\Im d\ identity\ opertors\ of\ \mathcal{K} \times \mathcal{K}\}$ given $\dim \mathcal{H} = \mathcal{K}$

(ii) Let $(\lambda_n)_{n\in\mathbb{N}} \subseteq W_{\wp_{\mathcal{K}}}(T)$ with $\lambda_n \to \lambda \in \mathbb{C}$. Since $\lambda_n \in W_{\wp_{\mathcal{K}}}(T)$ there exits $P \in \wp_{\mathcal{K}}$ and $x \in ran(P)$ such that $PAPx = \lambda_n x$.

Hence $\lambda f = lim_{n\to\infty}\lambda_n x = lim_{n\to\infty} PAPx = PAPx$ and consequently $\lambda \in W_{\wp_{\mathcal{K}}}(T)$.

(iii) Let $\lambda \in W(T)$ be given. Then there exists $x_o \in \mathcal{H}$ with $\|x_0\| = 1$ such that $\lambda = \langle Tx_0, x_0\rangle$. Now take $x_1, x_2 \dots \dots x_{\mathcal{K}-1} \in \mathcal{H}$ with $\|x_i\| = 1$ such that $x_i \perp x_j$, $i \neq j$, for $i, j = 0,1 \dots \mathcal{K} - 1$ and $x_i \perp Tx_0$ and $i = 1, \dots, \mathcal{K} - 1$.let T be orthogonal projection onto span $\{x_0, x_1 \dots, x_{\mathcal{K}-1}\}$ which is $\mathcal{K}$ dimensional subspace. Then, employing the fact that numerical range of a projection operator is convex,

$$PAPx_0 = PAx_0 = \langle Tx_0, x_0\rangle x_0 + \langle Tx_0, x_1\rangle x_1 + \cdots + \langle Tx_0, x_{\mathcal{K}-1}\rangle x_{\mathcal{K}-1}$$

$$= \langle Tx_0, x_0\rangle x_0$$

$$= \lambda x_0$$

i.e. λ Is an eigenvalue of PAP. Hence $\lambda \in W_{\wp_{\mathcal{K}}}(T)$

Now take $\lambda \in W_{\wp_{\mathcal{K}}}(T)$. Then there exist $\wp \in \wp_{\mathcal{K}}$ and $x \in ran(\wp)$ with $\|x\| = 1$ such that $PAPx = \lambda x$. Hence $\langle Tx, x\rangle = \langle PAPx, x\rangle = \langle \lambda x, x\rangle = \lambda$, implying $\lambda \in W(T)$.

■

Lemma 3.3.2. (***Karl Gustafson and K.M Rao*, 1995**). Let $T \in \mathbb{C}^2$ be normal operator in complex Hilbert space. Then $W(T) = \langle Tx, x\rangle$ is a line segment.

Proof

Let $T = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ where λ_1 and λ_2 are the eigenvalues of T and $x = (f, g)$. Then $\langle Tx, x\rangle = \lambda_1|f|^2 + \lambda_2|g|^2 = p\lambda_1 + (1-p)\lambda_2$ where $p = |f|^2$ and $|f|^2 + |g|^2 = 1$. Thus $W(T)$ is a set of convex combination of λ_1 and λ_2 and is the segment joining them. ■

Theorem: 3.3. 3 (***I.H.Sheth*, 1969**). Let $T \in \mathcal{B}(\mathcal{H})$ be an operator such that $T - \lambda I$ is normaloid for all complex values of λ. If $ST = T^*S$ for an arbitrary operator S for which

$0 \notin \overline{W(S)}$, then $T^* = T$

Proof:

Since $ST = T^*S$ and $0 \notin \overline{W(S)}$, then the spectrum is real. And also $Co\big(\sigma(S)\big) = W(S)$ for such an operator S. Hence the $\overline{W(S)}$ is real. ■

Remark 3.3.4. We note from the theorem above that if a normaloid operator $T - \lambda I$ with $\lambda \in \mathbb{C}$ is such that $ST = T^*S$ for some operator S whose numerical range property is $0 \notin \overline{W(T)}$ then $T^* = T$. Consequently $W(T)$ is real.

Theorem 3.3.5.$(\boldsymbol{S.K.Khasbardar\ and\ N.K.Thakare, 1978})$ Let $T, S \in \mathcal{B}(\mathcal{H})$ with T invertible such that

$S = TST^*$ with $0 \notin \overline{W(T)}$

If both T and T^{-1} are spectroloids, then T is unitary.

Proof:

If T is similar to a unitary operator then its $\mathfrak{r}(T) = 1$. Again if T is spectroloid we have

$\mathfrak{r}(T) = \mathcal{w}(T)$ and hence $W(T) \subset \bigoplus, unit\ disc$ on Hilbert complex plane. Since T^{-1} is also similar to a unitary operator we have $\mathfrak{r}(T^{-1}) = 1$ also;

$\mathfrak{r}(T^{-1}) = \mathcal{w}(T^{-1}) = 1$ and hence again $W(T^{-1}) \subset \bigoplus\ unit\ disc$ on Hilbert complex plane. ■

3.4 Numerical range of products of operators

In this section we study numerical range of product of operators though its information is scarce. Will also consider the numerical radius of operators that commute.

Theorem 3.4.1. $(\boldsymbol{Karl\ Gustafson\ and\ K.M\ Rao, 1995})$. Let T be a non-negative self adjoint operator and $TS = ST$. Then $W(TS) \subset W(T)W(S)$.

Proof:

$\langle TSx, x\rangle = \langle ST^{\frac{1}{2}}x, T^{\frac{1}{2}}x\rangle$. Where $T^{\frac{1}{2}}$ is the non-negative square root of T. Thus

$\langle TSx, x\rangle = \langle Sf, f\rangle \left\|T^{\frac{1}{2}}f\right\|^2 = \langle Sf, f\rangle\langle Tx, x\rangle$ where,

$$f = \frac{T^{\frac{1}{2}}x}{\left\|T^{\frac{1}{2}}x\right\|} \text{ With } T^{\frac{1}{2}}x \neq 0 \text{ and } \|f\| = 1.$$

Hence the self adjoint operator is non-negative.

Remark3.4.2. From the above theorem 3.4.1 we note that results on $W(TS)$ are harder to achieve. And thus we have an example that shows $w(TS) \geq w(T)w(S)$

Example 3.4.3. Let $T \in \mathbb{C}^4$ with $T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ then we have that,

$w(T) = \cos\left(\frac{\pi}{5}\right) = 0.80901699$ and also we have that $w(T^2) = w(T^3) = 0.5$ so that;

$$0.5 = w(T.T^2) > w(T).w(T^2) = 0.4045085$$

CHAPTER FOUR

THE NUMERICAL RANGE OF COMPACT OPERATORS

4.1 Introduction

In this chapter the properties of numerical range on compact operators were investigated. A bounded linear operator $T: X \rightarrow Y$ between Hilbert spaces is said to be compact if for every bounded sequence $\{x_n\} \subset X$, there exist subsequence $\{Tx_{n_k}\}$ of the sequence $\{Tx_n\}$ which converges in Y.

4.2 General Properties of Compact Operators

In this subsection we consider some general properties of compact operators. We note the following properties of compact operators which includes; their norm limit, commutativity, self-adjointness and lastly its spectral theory.

Proposition 4.2.1$(\boldsymbol{Halmos\ PR, 1982})$ Let T be bounded linear operator on Hilbert space $\mathcal{H}$.

(i) If S is a compact operator on Hilbert space $\mathcal{H}$, then ST and TS are compact.

(ii) If $\exists$ a bounded sequence T_n belonging to compact operators, so that $\|T - T_n\| \rightarrow 0$ as n$\rightarrow \infty$, then T is said to be compact.

(iii) Let T be a compact operator. Then T is a norm limit of a sequence of finite rank operators

(iv) T is compact if and only if T^* is compact.

Proofs

(i) If we let x_n be a bounded sequence, then Tx_n is also a bounded sequence and thus a bounded sequence STx_n has a convergent subsequence expressed as STx_{kn}.Therefore ST is compact. We also note that a bounded sequence Sx_n has a convergent subsequence Sx_{kn}. It follows that TSx_{nk} is convergent since T is continuous and hence TS is compact.

(ii) Let $x_n$be a bounded sequence and T_1 be a compact operator, then there exist a subsequence $x_{1,k}$ such that $T_1x_{1,k}$ Converges. Consequently let T_2 be compact then there exist a subsequence $x_{2,k}$ of $x_{1,k}$ such that $T_2x_{2,k}$converges and so forth. A class of nested subsequences $x_{n,k}$ is generated. If $g_k = x_{k,k}$, then $g_k$is eventually a subsequence of the nth subsequence $x_{n,k}$ so $T_n g_k$ converges as k→ ∞ for each $n \in \mathbb{N}$.Therefore Tg_k is a Cauchy sequence, and thus convergent.

This is expressed

$\|T(g_k - g_l)\| \leq \|Tg_k - T_m g_k\|$+$\|T_m(g_k - g_l)\|$+$\|T_m g_k - Tg_l\|$,

and holds for all m. Suppose M is a upper bound on the

$\|gk\|$, consequently

first and third terms are bounded by $M\|T - T_m\|$.

This is small provided m is chosen large enough. If we let m be large

Second term is small provided k, l are large enough

(iii) If $e_1 e_2$ ... is an orthonormal basis and P_n a orthogonal projection onto the span of the first n basis vectors, and $Q_n = Id - P_n$, then $\|Q_n Tx\|$ is a non-increasing function of n.So $\|Q_n T\|$ is non-increasing in n. If $\|Q_n T\| = \|P_n T - T\| \to 0$ hence the statement is justified. So assume, for a contradiction, that $\|Q_n T\| \geq c$ for all n. Choose x_n with $\|x_n\| = 1$ such that $\|Q_n Tx_n\| \geq {}^{c}/_{2}$ for each n. By compactness of T, there is a subsequence such that $Tx_{kn} \to g$ for some g. Then, $\|Q_n Tx_{kn}\| \leq$

$\|Q_{kn}g\| + \|Q_{kn}(g - Tx_{kn})\| \le \|Q_{kn}g\| + \|g - Tx_{kn}\|$ (Since$\|Q_{kn}\| = 1$) and both the terms on the right hand side converge to zero, which is the desired contradiction.

(iv) This follows from parts (ii) and (iii), and from the identity $\|T\| = \|T^*\|$ for all bounded linear transformation T ■

Remark. 4.2.2 We note the following important theorem is a direct analogue of the spectral theorem for symmetric matrices.

4.3 Spectral theorem for compact operators

Theorem 4.3.1($\boldsymbol{Halmos\ PR, 1982}$): Let T be a compact selfadjoint operator on $\mathcal{H}$.Then there is an orthonormal basis e_1, e_2 ....of $\mathcal{H}$ consisting of eigenvalues of T. Thus $T\,e_i = \lambda_i e_i$, and we have

$\lambda_i \in \mathbb{R}$ and $\lambda_i \to 0$ as $i \to \infty$ we will consider the following steps.

Proof

(i) We start by showing that $\|T\| = sup_{\|x\|=1}|\langle Tx, x\rangle|$.

To see this we use characterization, let $M = sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$. For $x \in \mathcal{H}$ with $\|x\| = 1$,

$|\langle Tx, x\rangle| \le \|Tx\|\|x\| \le \|T\|\|x\|^2 = \|T\|$. This show that $M \le \|T\|$ for all $x \in \mathcal{H}$.

To prove $M \ge \|T\|$ we use selfadjointness of operator T,

$4Re\langle Tx, y\rangle = (\langle T(x + y), x + y\rangle - \langle T(x - y), x - y\rangle)$.

Then, we get $4|Re\langle Tx, y\rangle| \le M(\|x + y\|^2 + \|x - y\|^2)$.

And the ‘parallelogram law’ gives $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) = 4$ so$|Re(Tx, y)| \le M$. Replacing y by $e^{i\theta}$y we can make $|Re(Tx, y)| = |(Tx, y)| \le M$ hence $\ge \|T\|$. Thus the equality holds that is, $\|T\| = sup_{\|x\|=1}|\langle Tx, x\rangle|$.

(ii) For some eigenvector x corresponding to operator T we show that the quantity on the right hand side takes a maximum value.

However for operator $T = 0$, then in this case the theorem is trivial or $|(Tx, x)| > 0$ for all

$x \in \mathcal{H}$ when $\|x\| = 1$. And by substituting T with $-T$ we can assume that there exists x with $\langle Tx, x\rangle > 0$ N.B by selfadjointness the numerical range is real. Considering the problem of maximizing $\langle Tx, x\rangle$ for all $x \in \mathcal{H}$ has a sup $\mathcal{u} = \|T\| > 0$ this can take a sequence

$x_n, x_n = 1$, with $\langle Tx, x\rangle \to \mathcal{u}$ then

$\|(T - \mathcal{u})x_n\| \to 0$. Squaring the left hand side of the equation and computing we have

$$0 \le \|(T - \mathcal{u})x_n\|^2 = \|Tx_n\|^2 - 2\mathcal{u}(Tx_n, x_n) + \mathcal{u}^2$$

$$\le 2\mathcal{u}(\mathcal{u} - \langle Tx_n, x_n\rangle) \to 0.$$

Hence the proof that $\mathcal{u} = \|T\| > 0$. Exploiting the compactness of T: the sequence (Tx_n) has a subsequence converging, say to $\mathcal{u}x$. Considering the subsequence we may assume that the sequence (Tx_n) itself converges. Then x_n converges to x, since $\|x_n - x\| \le \mathcal{u}^{-1}(\|(T - \mathcal{u})x_n\| + \|Tx_n - \mathcal{u}x\|) \to 0$. By Continuity of T, $Tx = lim_{n\to 0} T(x_n) = \mathcal{u}x$. Thus the eigenvector x of T exists.

(iii) Eigenspaces of T corresponding to distinct eigenvalues are orthogonal.

If $Tv = \lambda v$ and $Ty = \mathcal{u}y$, then we have,

$(v, y) = \lambda^{-1}(Tv, y) = \lambda^{-1}(v, Ty) = \mathcal{u}\lambda^{-1}(v, y)$. So $(v, y) = 0$ unless $\lambda = \mathcal{u}$.

(iv) The operator T restricts to a compact self-adjoint operator $T_{/V^{\perp}}$ whenever V is an eigenspace, or direct sum of eigenspaces.

If selfadjoint operator T preserves a subspace i.e. $(Tv) \in V$ for all $v \in V$, then it also preserves the orthogonal compliment, since $\langle Tx, v\rangle = \langle x, Tv\rangle$. Thus the operator T

preserves each eigenspace and thus T restrict to an operator on the orthogonal compliment of all spaces.

(v) Thus the direct sum of eigenspaces must be the whole spaces.

Lastly, it's clear that the orthogonal compliment of all the eigenspaces must be the null space, or else by the above, the operator T restricts to it and has eigenvector there.

■

4.4 The numerical range and spectral properties on compact operators

In this subsection we start with an example to show that the numerical range of compact operator is not necessarily closed.

Example 4.4.1: The numerical range of a compact operator isn't always closed for instance

$W(T) = (0,1]$, that is $0 \in \overline{W(T)}$ but $0 \notin W(T)$.

Theorem 4.4.2$(\boldsymbol{De\ Barra, et\ al, 1972})$. If T is compact and $0 \in W(T)$, then $W(T)$ is closed.

Proof:

If $0 \in W(T)$, then $\langle Tx, x\rangle \in W(T)$ for every unit vector x reason if $\|x\| = 1$ and

$0 \leq t \leq 1$ then

$\langle T(tx), x\rangle = t\langle Tx, x\rangle = t\langle Tx, x\rangle + (1-t).0 \in W(T)$

Next is to show that if compactness exist then the numerical range is continuous on bounded sets. If a sequence $\{x_n\}$ is bounded and weakly convergent to x, then

$|\langle Tx_n, x_n\rangle - \langle Tx, x\rangle| \leq |\langle Tx_n, x_n\rangle - \langle Tx, x_n\rangle| + |\langle Tx, x_n\rangle - \langle Tx, x\rangle|$

The first summand tends to 0 because $\{Tx_n\}$ is strongly convergent and the second summand tends to 0 becouse $x_n \to x$ weakly. ■

Theorem 4.4.3. Let $T \in \mathcal{B}(\mathcal{H})$ be compact. Then $\sigma(T) \subset W(T)$ under any one of the following conditions

(i) $0 \notin \sigma(T)$

(ii) $0 \in W(T)$

Proof

We first note that in general $\sigma(T) \subset \overline{W(T)}$. However for T compact, $0 \notin \sigma(T)$ implies $\sigma_p(T) = \sigma(T)$. Thus every non-zero element of $\sigma(T)$ is an eigenvalue.

But $\sigma_p(T) \subset W(T)$

(i) Indeed if $\lambda \neq 0 \in \sigma(T)$ .Then $Tx = \lambda x$ and thus for x with $\|x\| = 1$,

$W(T) = \langle Tx, x\rangle = \langle \lambda x, x\rangle = \lambda\langle x, x\rangle = \lambda\|x\|^2 = \lambda$ i.e. $\lambda \in W(T)$

(ii) Now $0 \in W(T)$ implies $W(T)$ is closed by theorem 4.4.2 above.

Thus $\overline{W(T)} = W(T)$ and hence $\sigma(T) \subseteq W(T)$.

The following corollary is immediate ■

Corollary 4.4.4:

Let $T \in \mathcal{B}(\mathcal{H})$ be a compact operator then $\sigma(T)$ and $W(T)$ are spectral sets under any of the conditions of theorem 4.4.3 above. ■

4.5. Numerical range of a product of compact operators

Theorem 4.5.1: $(\boldsymbol{Wafula, A\ and\ J. Khalagai, 2019})$ If $T, S \in \mathcal{B}(\mathcal{H})$ are self-ad joint λ commuting operators then $W(TS)$ and $W(ST)$ are real.

Proof:

Let $u \in W(TS)$. Then $u = \langle TSx, x\rangle = \langle Sx, Tx\rangle$. But since

$W(TS) = W(\lambda ST) = \lambda W(ST)$, we must have that

$\lambda\langle STx, x\rangle = \lambda(Tx, Sx) = \lambda^{-}\langle Sx, Tx\rangle = \lambda u^{-} = u$.

For T and S self adjoint $\lambda = \pm 1$ so that $u = \pm u^{-}$ ■

Corollary 4.5.2: Let $T, S \in \mathcal{B}(\mathcal{H})$ be such that T is compact then we have:

(i) $0 \notin \sigma(TS)$ implies $\sigma(TS) \subset W(TS)$

(ii) $0 \in W(TS)$ implies $\sigma(TS) \subset W(TS)$

Proof

(i) We first note that T is compact, implies TS is compact now by part (i) of theorem 4.4.3 above $\sigma(TS) = \sigma_p(TS) \subset W(TS)$.

(ii)Similarly by part (ii) of theorem 4.4.3 above we have $0 \in W(TS)$ implies $W(TS)$ is closed. Hence $\sigma(TS) \subset W(TS) = \overline{W(TS).}$ ■

Corollary 4.5.3: Let $T, S \in \mathcal{B}(\mathcal{H})$ be such that T is compact if $0 \notin \sigma(TS)$ and $0 \in W(ST)$ with either T or S invertible then we have that $\sigma(TS) = \sigma(ST) \subset W(TS)$ and $\sigma(TS) = \sigma(ST) \subset W(ST)$.

Proof

For definiteness we assume that T is invertible then we have $ST = T^{-1}(TS)T$ which implies TS and ST are similar. Hence $\sigma(TS) = \sigma(ST)$ .Thus by **Corollary** 4.5.2 above we have $\sigma(TS) = \sigma(ST) \subset W(TS)$ and $\sigma(TS) = \sigma(ST) \subset W(ST)$.

Remark 4.5.4. We note that for an operator T we have that $W(T)$ is real does not necessarily imply $\sigma(T)$ is also real. However in view of theorem 4.5.1 above we have the following result.

Theorem 4.5.5: Let $T, S \in \mathcal{B}(\mathcal{H})$ be self adjoint $\lambda-$ commuting operators with T compact. If

$0 \notin \sigma(TS)$ and $0 \in W(ST)$ then we have that $\sigma(TS)$ and $\sigma(ST)$ are real.

Proof:

We first note that since T is compact both TS and ST are compact. By theorem 4.5.1 above T and S are $\lambda-$commuting self adjoint operators implies $W(TS)$ and $W(ST)$ are real.

Now $0 \notin \sigma(TS)$ implies $\sigma(TS) = \sigma_p(TS) \subset W(TS)$. Hence $\sigma(TS)$ is real.

Also $0 \in W(ST)$ implies $W(ST) = \overline{W(ST)}$

Thus $\sigma(ST) \subset W(ST)$. Hence $\sigma(ST)$ is also real. ■

Remark 4.5.6.

We note that in theorem 3.2.5 above if the operator T is compact then we can relax the condition $0 \notin \overline{W(T)}$ and still same proof carries through as the result shows below.

Theorem 4.5.7: Let $T \in \mathcal{B}(\mathcal{H})$ be compact such that $0 \in W(T)$ and $0 \notin \sigma(T)$. Then for any other operator S we have $\sigma(T^{-1}S) \subset \overline{W(S)} - W(T)$.

Proof

We first note that $0 \notin \sigma(T)$ implies T^{-1} exists and the identity

$T^{-1}S - \lambda = T^{-1}(S - \lambda T)$ shows that if $\lambda \in \sigma(T^{-1}S)$ then $0 \in \sigma(S - \lambda T)$. Thus we have

$0 \in W\overline{(S - \lambda T)} \subset \overline{W(S)} - \lambda W(T)$.

Thus $\lambda \in \overline{W(S)} - W(T)$ .Hence $\sigma(T^{-1}S) \subset \overline{W(S)} - W(T)$. ■

Corollary 4.5.8. If in theorem 4.5.7 above the operator S is also compact with $0 \in W(T)$ then we have, $\sigma(T^{-1}S) \subset W(S) - W(T)$

Proof

In this case $0 \in W(S)$ implies $W(S)$ is closed and hence $(S) = \overline{W(S)}$. Hence the results follow.

■

CHAPTER FIVE

THE INVARIANCE OF NUMERICAL RANGE UNDER PARTIAL ISOMETRIES

5.1 Introduction

The invariance of numerical range under partial isometries is seen in the subclasses whose inclusions is as follows $\{unitary\} \subset \{isometry\} \subset \{partial\ isometry\}$ and $\{unitary\} \subset \{co-isometry\} \subset \{partial\ isometry\}$

5.2 General Properties of Partial Isometries

Theorem 5.2.1 $(\boldsymbol{Takayuki\ Furuta, 2001})$**.** An operator T is unitary on a Hilbert space $\mathcal{H}$ if and only if $T^*T = TT^* = I$

Proof:

T if and only if it is unitary, it is an operator of isometry from $\mathcal{H}$ onto $\mathcal{H}$, that is $T^*T = I$ for all $x \in \mathcal{H}$, then $\exists\ y \in \mathcal{H}$ such that $Ty = x$ and $T^*x = T^*Ty = y$. Hence $\|T^*x\| = \|y\| = \|Ty\| = \|x\|$ .Thus T^* is an isometry and $T^*T = (T^*)^*T^* = I$

Conversely if the operator T is unitary that is $T^*T = TT^* = I$. Then T is isometry and for any $x \in \mathcal{H}$, $x = TT^*x \in R(T)$, where $R(T)$ means the range of T, so T is an operator of isometry from $\mathcal{H}$ onto $\mathcal{H}$. ■

Theorem 5.2.2$(\boldsymbol{Takayuki\ Furuta, 2001})$**.** The operator T on the Hilbert space is an isometry operator if and only if the $T^*T = I$.

Proof:

Since we have T as an isometry then $\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \langle x, x\rangle$ for all $x \in \mathcal{H}$ and thus $T^*T = I$ .Conversely if $T^*T = I$ then it implies that,

$\|Tx\|^2 = \langle T^*Tx, x\rangle = \langle x, x\rangle = \|x\|^2 = I$. Hence the operator T is an isometry.

Theorem 5.2.3 ($\boldsymbol{Takayuki\ Furuta, 2001}$): Let T be an operator of partial isometry on a Hilbert space, M an initial space and N final space, then the following statements hold.

Proof:

(a) $TP_M = T$ and $T^*T = P_m$ for all $x \in \mathcal{H}$, let $x = P_M x \oplus z$ for $z \in M^\perp$, and

$Tx = TP_M x \oplus Tz = TP_M$ Since $Tz = 0$.

Hence $T = TP_M$. As $\langle Tx, Ty\rangle = \langle x, y\rangle$ for $x, y \in M$ and $P_M x, P_M y \in M$ for all $x, y \in \mathcal{H}$.

$\langle T^*Tx, y\rangle = \langle Tx, Ty\rangle = \langle TP_M x, TP_M y\rangle = \langle P_M x, P_M y\rangle = \langle P_M x, y\rangle$.

Hence $T^*T = P_M$.

(b) N is closed subspace of $\mathcal{H}$ since $N = R(T) = TR(P_M) = TM$, for all $x \in \overline{N}$, there exists a sequence $\{y_n\} \subset M$ such that $Ty_n \to x$ and

$\|y_m - y_n\| = \|Ty_m - Ty_n\| \to 0$ as $m, n \to \infty$ hence by completeness of Hilbert space, $\exists\ y \in \mathcal{H}$ such that $y_n \to y$, and $Ty_n \to Ty$ implies $x = Ty \in N$, thus $\overline{N} = N$

(c) With the initial space N, T^* is a partial isometry with final space M, is $T^*P_N = T^*$ and $TT^* = P_N$

For all $x \in N$ there exist $y \in M$ such that $Ty = x$ and $\|x\| = \|y\|$, and

$T^*x = T^*x = T^*Ty = P_m y = y$, so that $\|T^*x\| = \|x\|$ for all$x \in N^\perp$.

Since $Ty \in N$ for any $y \in \mathcal{H}$, so that $T^*x = 0$.Thus T^* is a partial with initial space N and the final space M because,

$R(T^*) = T^*N = T^*R(T) = T^*T\mathcal{H} = P_m\mathcal{H} = M$.

$T^*P_N = T^*$ and $TT^* = P_N$ follow from (a) by substituting T by T^* and M by N. ■

THEOREM 5.2.4 $(\boldsymbol{Takayuki\ Furuta}, \mathbf{2001})$**.** Let $T \in \mathcal{B}(\mathcal{H})$.Then the following are equivalent statements.

(a) T is a partial isometry operator

(b) $TT^*T = T$

(c) T^*T is a projection operator.

Proof:

$(a) \Longrightarrow (b)$: Since we have $TP_M = T$ and $T^*T = P_m$ then we have,

$TT^*T = TP_M = T$

$(b) \Longrightarrow (c)$: Multiplying T^* on both side of (b) we have $T^*TT^*T = T^*T$ and by (b) then T^*T is a projection operator

$(c) \Rightarrow (a)$: Let $T^*T = P_M$ for all $x \in \mathcal{H}$ then,

$\|Tx\|^2 = \langle T^*Tx, x\rangle = \langle P_M x, x\rangle = \|P_M x\|^2$ thus $\|Tx\| = \|x\|$ for all $x \in M$ and $Tx = 0$ for all $x \in M^{\perp}$.Hence evidence of the equivalence relation between (a), (b) and (c) holds. ■

Proposition 5.2.5. $(\boldsymbol{Joel\ H\ Shapiro}, \mathbf{2004})$. For any operator $T \in \mathcal{B}(\mathcal{H})$, $W(T)$ is invariant under unitary similarity

Proof:

If $S \in \mathcal{B}(\mathcal{H})$ is unitary that is $SS^* = S^*S = I$ .this means $\|S\| = 1$ and $S^* = S^{-1}$, then we have $\langle S^*TSx, x\rangle = \langle TSx, Sx\rangle$, Sx is a unit vector norm and hence,

$\langle Sx, Sx\rangle = \langle SS^*x, x\rangle = \langle x, x\rangle = I$, we have our results as

$W(T) = \langle TSx, Sx\rangle = \langle S^*TSx, x\rangle = W(S^*TS)$ ■

Proposition 5.2.6 $(\boldsymbol{M.Newman, 1982})$. For any operator $T \in \mathcal{B}(\mathcal{H})$ the numerical radius is invariant under unitary similarity. Thus any unitary operator S, we have that, $w(S^*TS) = w(T)$

Proof:

For all unit vectors $x \in \mathcal{H}$ we have that,

$w(S^*TS) = sup|(S^*TSx, x: \|x\| = 1)|$

$= sup|(TSx, Sx: \|x\| = 1)|$. For $\langle Sx, Sx\rangle = \langle x, x\rangle = 1$

We have sup$|(Tx, x: \|x\| = 1)|$. Hence $w(T)$ =sup$|(Tx, x: \|x\| = 1)|$.

Thus we conclude that $w(S^*TS) = w(T)$. ■

Remark 5.2.7. Since the numerical radius is invariant under unitary operator and we know that $\mathfrak{r}(T) \leq w(T)$ which is a result of $\sigma(T) \subset W(T)$ then the spectral radius will also be invariant under the same unitary operator.

Proposition 5.2.8$(\boldsymbol{M.Newman, 1982})$: For any operator $T \in \mathcal{B}(\mathcal{H})$ the spectral radius is invariant under unitary similarity. Thus for any unitary operator S, we have that, $\mathfrak{r}(S^*TS) = \mathfrak{r}(T)$

Proof:

For all unit vectors $y \in \mathcal{H}$ we have that

$$\mathfrak{r}(S^*TS) = sup\{\langle S^*TSy, y: \|y\| = 1\rangle\} = sup\{|\lambda|: \lambda \in \sigma(T)\ for\ (\lambda I - T) \neq 0\}$$

$$= sup\{TSy, Sy: \|y\| = 1\}.$$

For $\langle Sy, Sy\rangle = \langle y, y\rangle = 1 = sup\{Ty, y\} = \mathfrak{r}(T) = sup\{|\lambda|: \lambda \in \sigma(T)\}$.

Thus $(S^*TS) = \mathfrak{r}(T)$.

Hence the spectral radius is invariant under unitary operator S ■

5.3 Invariance of numerical range of subclasses of partial isometries

In this subsection we deal mainly with isometries and co isometries since the case of unitary operators has already been covered in subsection 5.2.

Lemma 5.3.1: Let an operator S be an isometry, then for each unit vector x we have $\|Sx\| = 1$.

Proof:

If a unit vector x is of the form Sx then, $\|Sx\|^2 = \langle Sx, Sx\rangle = \langle x, S^*Sx\rangle$.

But $S^*S = 1 = \langle x, x\rangle$.Since$\|x\|^2 = 1$ ■

Theorem 5.3.2. $(\boldsymbol{Wafula, A\ and\ J. Khalagai, 2010})$**:** Let $T \in \mathcal{B}(\mathcal{H})$ and S an isometry, then $W(S^*TS) \subset W(T)$

Proof

Suppose $\lambda \in W(S^*TS)$, then we have $\lambda = \langle S^*TSx, x\rangle$ for all unit vector $x = (TSx, Sx)$. Thus $\|Ax\|^2 = \langle TSx, Sx\rangle = \langle x, S^*S\rangle = \langle x, x\rangle = 1$. Hence λ belongs to $W(T)$. ■

Theorem 5.3.3. $(\boldsymbol{Wafula, A\ and\ J. Khalagai, 2010})$**.** If S is an isometry and every unit vector of $\mathcal{H}$ is of the form Sx, then

$W(S^*TS) = W(T)$ for any bounded operators T on $\mathcal{H}$.

Proof

Suppose $\lambda \in W(T)$ and $\lambda = \langle Tx, x\rangle$ for all unit vectors x in $\mathcal{H} = \langle TSy; Sy\rangle$ for some unit vector y in $\mathcal{H} = \langle S^*TSx, x\rangle \in W(S^*TS)$. Hence the results,

$W(T) \in W(S^*TS)$ is achieved. Thus from previous theorem, we have that

$W(S^*TS) = W(T)$. ■

Corollary 5.3.4. $(\boldsymbol{Wafula, A\ and\ J. Khalagai, 2010})$**.** If S is a co-isometry such that each unit vector is of the form Sx for some $x \in \mathcal{H}$, then for any bounded operator T, we have that $W(STS^*) = W(T)$. ■

Theorem 5.3.5. $(\boldsymbol{Wafula, A\ and\ J. Khalagai, 2010})$**.** If a partial isometry S is either injective or has a dense range and each unit vector is of the form Sx, then $W(S^*TS) = W(T)$ or $W(STS^*) = W(T)$ for all bounded operators T

Proof

Assuming that an operator S is partial isometry then $SS^*S = S$,then have

$S = SS^*S \Leftrightarrow S - SS^*S = 0 \Leftrightarrow S(I - S^*S) = 0$ but since $S \neq 0$ we have;

$(I - SS^*) = 0$.If S is one-one then,

$I - S^*S = 0 \Leftrightarrow S^*S = I$. Hence S is an isometry

If S has a dense range $I - SS^* = 0 \Leftrightarrow SS^* = I$ then S is co-isometry. ■

5.4 Numerical range of product of partial isometries

Lemma 5.4.1 $(\boldsymbol{Karl\ Gustafson\ and\ K. M\ Rao, 1995})$. Let $T \in \mathcal{B}(\mathcal{H})$ and S be an isometry such that $ST = TS$. Then $w(ST) \leq w(T)$

Proof:

Let $S^*S = I$ and $\langle STx, x\rangle = \langle S^*STx, x\rangle = \langle TSx, Sx\rangle$. Considering the restriction to the range of operator S which is closed, we notice that on the range of S, S is unitary, because $S^*S = I$ and for all $x = Sg \in R(S)$, we have,

$SS^*x = SS^*Sg = Sg = x$. So $SS^* = I$ on $R(S)$. Since S is unitary on $R(S)$, we conclude that $w(ST) \leq w(T)$ on $R(S)$ ■

Corollary 5.4.2 $(\boldsymbol{Karl\ Gustafson\ and\ K.M\ Rao, 1995})$. Let S be a unitary operator that commutes with another operator T.Then

$$w(ST) \leq w(T).$$

Proof:

This follows trivially from lemma 5.4.1 above since every unitary operator is an isometry. ■

CHAPTER SIX

SUMMARY AND RECOMMENDATION

6.1 Summary

The aim of this research was to determine the location of numerical range and its role on properties of some specific classes of operators, the first chapter covered background information, definitions and statement of the problem, the second chapter reviewed the literature related to the study. However the main results were obtained from the classes of operators which have been covered from the third chapter onwards.

In chapter three the study covered some general properties of numerical range of operators, in which the main results were containment of the point spectrum in the numerical range that is in lemma 3.2.2, the containment of the spectrum in the closure of the numerical range in theorem 3.2.3 and an extension to the theorem 3.2.3 where the spectra of product of an invertible operator with any other operator is contained in closure of the difference of their numerical ranges that is in theorem 3.2.5. The study further investigated the behavior of the numerical range of projection operators, self adjoint operators and normal operators in which all their numerical ranges lied on the real axis, their results are stated in theorem 3..3.1, theorem 3.3.3, theorem 3.3.5, respectively.

The spectral properties of compact operators in chapter four theorem 4.3.2 discusses the existence of an orthonormal basis which consists of eigenvalues for compact self adjoint operators. In theorem 4.4.2 whenever $0 \in W(T)$ then the numerical range is closed. In this study in theorem 4.4.3 the spectrum of a compact operator is contained in the numerical range of the operator unlike the condition given in theorem 3.2.3, the study further showed that the spectra of products of a compact operator in theorem

4..5.7 is contained in the difference of the closure of numerical range and the spectrum of compact operator is contained in the numerical range.

In chapter five proposition 5.2.5 shows the numerical range of a bounded linear operator is invariant under unitary similarity, theorem 5.3.10 and theorem 5.3.11 shows that the numerical range of a bounded linear operator is invariant under an isometry operator.

Thus, the main results of the numerical range of the specific classes of operators has been obtained in this study and are in accordance to the stated objectives.

6.2 Recommendation

The recommendation for further studies are as follows.

(i) In chapter three it is a regret that the location of the numerical range of subclasses of spectroloid operators has been limited to projections, selfadjoint operators and normal operators but for normaloid operators and spectroloid operators the location is not yet exploited.

(ii) In chapter four the location of the numerical range of product of compact operators have been identified only where they are λ commuting, while the location of the numerical range of compact operator itself is not yet identified. This requires further investigation.

(iii) In chapter five the location of the numerical range of unitary operator has been identified, while that of isometry and partial isometry is not well shown. Further studies are required on invariance of numerical range for some classes of operators.

(iv) In this thesis we have carried out the study as stated above on three classes of operators namely spectroloid, compact and partial isometries. Further studies can be undertaken using different sets of classes of operators.

REFERENCES

Dada, W., Erkurşun, N., & Kerner, J. (2016). Numerical range with respect to a family of projections. ArXiv preprint arXiv: 1608.04218.

De Barra, G., Giles, J. R., & Sims, B. (1972). On the numerical range of compact operators on Hilbert spaces. Journal of the London Mathematical Society, 2(4), 704-706.

Embry, M. (1970). The numerical range of an operator. Pacific Journal of Mathematics, 32(3), 647-650.

Furuta, T. (2001). Invitation to linear operators: From matrices to bounded linear operators on a Hilbert space. CRC Press.

Goldberg, M., & Tadmor, E. (1982). On the numerical radius and its applications. Linear Algebra and Its Applications, 42, 263-284.

Gustafson, K. E., & Rao, D. K. (1995). Numerical range. In Numerical range (pp. 1-26). Springer, New York, NY.

Halmos, P. (1982). A Hilbert space problem Book. 2nd springler-Verlag, New York.

Hausdorff F (1919),Der wertrorrant einer bilinear form. Math Z 3,314-316

Khasbardar, S. K. (1978). Spectroloid operators, similarity and related results.

Meng C.H(1957) A condition that a normal operator has closed numerical range. Proc Am Math Soc 8, 85-88

Shapiro, J. H. (2004). Notes on the numerical range. Michigan State University, East Lansing, MI 48824-1027, USA.

Sheth, I. H. (1969). Normaloid operators. Pacific j. Math, 28(3), 675-680.

Toeplitz O. (1918), Das algebraische Analogon zu einemen satz von Fejer, Math Z, 187-197

U.N Singh and K.Mangla. Operators with inverses similar to their adjoints. Proc Am Soc V38 P258-260m

Wafula A.W and J.M khalagai. June 2019. On numerical range of $\lambda -$ commuting operators, International journal of creative research and studies, Issn-0249-4655.

Wafula, A. W., & Khalagai, J. M. (2010). Invariance of numerical range under isometric transformation. African Journal of Mathematics and Computer Science Research, 3(2), 036-037.

Williams J.P and Criminis T (1974).On the numerical radius of a linear operator. Am Math 74,832-833

Williams J.P, (1967) .Spectra of products and Numerical range. J Math Am Appl 17, 2-4-2200

Williams J.P, (1969). Similarity and numerical range .J. Math AM. Appl26, 307-314.

APPENDICES

APPENDIX I: SIMILARITY INDEX/ANTI-PLAGIARISM REPORT

Turnitin https://www.turnitin.com/newreport_classic.asp?lang=en_us&o...

Document Viewer

Turnitin Originality Report

Processed on: 25-Feb-2021 08:55 EAT
ID: 1517702072
Word Count: 6828
Submitted: 1

ON NUMERICAL RANGE OF SOME CLASSES OF OPERATO... By Kipruto Emmanuel

Similarity Index	Similarity by Source	
18%	Internet Sources:	10%
	Publications:	14%
	Student Papers:	6%

include quoted include bibliography excluding matches < 5 words mode:
quickview (classic) report Change mode print refresh download

1% match (publications)
Karl E. Gustafson, Duggirala K. M. Rao. "Numerical Range", Springer Science and Business Media LLC, 1997

1% match (publications)
Paul R. Halmos. "A Hilbert Space Problem Book", Springer Science and Business Media LLC, 1982

1% match (publications)
Harkrishan Lal Vasudeva. "Elements of Hilbert Spaces and Operator Theory", Springer Science and Business Media LLC, 2017

1% match (Internet from 08-Feb-2021)
https://www.coursehero.com/file/41457748/Mutegi-Thesis-Revised-AO-July-2018doc/

1% match (publications)
Furuta, . "Fundamental Properties of Bounded Linear Operators", Invitation to Linear Operators, 2001.

1% match ()
http://seisummit.org

1% match (student papers from 15-Oct-2018)
Submitted to Kisii University on 2018-10-15

<1% match (publications)
Daniel Alpay, Fabrizio Colombo, Irene Sabadini. "Slice Hyperholomorphic Schur

1 of 18 2/25/2021, 9:13 AM