ON ALMOST SIMILARITY AND SOME CLASSES OF OPERATORS IN HILBERT SPACES

BY

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DECLARATION

Declaration by the candidate

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DEDICATION

This thesis is dedicated to my beloved mother, Mary Kangogo.

ABSTRACT

Similarity is a recent area of study on classes of operators in Hilbert spaces. The study of operators, under similarity and quasisimilarity concepts, motivated researchers to extend their research to almost similarity property which is undergoing current research. Almost similarity has been shown to be an equivalence relation and to preserve nullity and co-nullity of operators. Though similarity preserves nontrivial subspaces and quasisimilarity preserves hyperinvariant subspaces of operators, there is scanty literature linking such conclusion to almost similarity. Properties of almost similarity on some classes of operators, namely, partial isometries, θ-operators, posinormal operators among others and conditions under which almost similarity gives equality of spectra remain open for more research. The main purpose of this research was to investigate almost similarity properties on partial isometries, θ-operators, posinormal operators and conditions yielding equality of spectra. Comparative and analytic approaches were used by considering known results on similarity and quasisimilarity concepts. Among other results, unitary equivalence of both θ-operators and posinormal operators under isometry and co-isometry properties were established. Results from the study are fundamental as they will bring about more understanding on properties of operators, which is the basis for those applying these operators in quantum mechanics, spectral analysis of functions and unitary group representation.

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INDEX OF NOTATIONS AND TERMINOLOGIES

The following notations are used this thesis:

$\mathbb{C}$: Set of complex numbers.

$\mathbb{R}$: Set of real numbers.

H, K: Hilbert spaces over the complex scalars $\mathbb{C}$.

$\mathfrak{B}(H)$: Set of bounded linear operators on H.

$\mathcal{R}$: The real part of a complex number.

$\langle x, y \rangle$: Inner product of vectors x and y in a Hilbert space.

$\|x\|$: Norm of a vector x.

$A, B, \mathcal{T}, \mathcal{S}, \mathcal{V}, \mathcal{W}$: Operators acting on a Hilbert space.

$\mathcal{T}^*$: Adjoint of $\mathcal{T}$.

$Ker\ (\mathcal{T})$: Kernel of $\mathcal{T}$.

$Ker\ (\mathcal{T}) \equiv N(\mathcal{S})$: Null space of $\mathcal{S}$.

$Ran\ (\mathcal{T})$: Range of $\mathcal{T}$.

$\|\mathcal{T}\|$: Norm of $\mathcal{T}$.

$|\mathcal{T}|$: Absolute value of $\mathcal{T}$.

$\sigma(\mathcal{T})$: Spectrum of $\mathcal{T}$.

$W(\mathcal{T})$: Numerical range of $\mathcal{T}$.

$\omega(\mathcal{T})$: Numerical radius of $\mathcal{T}$.

$\rho(\mathcal{T})$: Resolvent set of $\mathcal{T}$.

$\{\mathcal{T}\}'$: Commutator of $\mathcal{T}$.

$\sigma_{po}(\mathcal{S})$: Posispectrum of $\mathcal{S}$.

$\sigma_{\pi}(\mathcal{S}^*)$: Approximate point spectrum of adjoint of $\mathcal{S}$.

$\bigoplus$: Direct sum.

$\forall$: For all.

$a.s$: Almost similar.

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CHAPTER ONE

INTRODUCTION

1.1 Background

A Hilbert Space is a mathematical space named after David Hilbert. Hilbert spaces by David Hilbert, (20^{th}century), generalize the notion of a Euclidean space, extending the methods of vector algebra and calculus from the two dimensional Euclidean plane and three dimensional spaces to spaces with any finite or infinite dimensions. Prior to the development of Hilbert spaces in first decade of the 20^{th}century, Physicists and Mathematicians in 18^{th} and 19^{th} centuries used a generalized Euclidean idea of a space (abstract linear space), whose elements can be multiplied and added together by a scalar. The development of Hilbert Space was sequential. The first establishment arose on study of integral equations by David Hilbert and Erhard Schmidt. They showed that two square integrable real-valued functions f and g of an interval $[a, b]$ contain an inner product given by $\langle f, g\rangle = \int_a^b f(x)g(x)dx$. This was followed by Lesbesgue Integral, an extension from Riemann Integral by Heine (1904), which made it possible to integrate a much broader class of functions. Frigyes and Ernst (1907) proved that the space L^2 of square integrable functions is a complete metric space. It was further advanced to more general spaces by trigonometric series results, studied by Joseph Fourier, Friedrich Bessel and Marc-Antoine Parseval. Further basics were proved in the early 20^{th} century by other scholars. John-Von Neumann used abstract Hilbert Space in his study on unbounded self-adjoint (Hermittian) operators and foundations of quantum mechanics and demonstrated axiomatic results. This introduced Hilbert Space to many scholars since it offers the best mathematical formulations of quantum mechanics. Quantum mechanical systems are vectors in a certain Hilbert Space classified easily with operators like self-adjoint, unitary and normal. Operators are frequently used to perform a specific mathematical operation on another function. They are also used in Physics and Chemistry to simplify complex equations like the Hamiltonian operator used to solve the Schrodinger wave equations to give energy associated with a given wave function. Hermann (1909) developed the classical Weyl theorem, a new concept in operator theory and proved that, for two given self-adjoint operators A and B, then the spectrum of A and $A + B$ have equal limit points if B is a compact operator. He further showed that the essential spectra of a self-adjoint operator is invariant under perturbations and that compact perturbation of an operator is in some sense small and summarized that Fredholm index is a topological quantity. According to Von Neumann (1935), if the spectra of self-adjoint operators A and B have the same limit points and given a compact operator K, then $A + K$ and B are unitarily equivalent. Generally, significant knowledge contribution has been done by varied scholars on relations among operators in Hilbert spaces which include similarity and quasimilarity but scantly on almost similarity. Jibril (1996) introduced this property of almost similarity. According to him, two operators $A, B \in \mathfrak{B}(H)$ are almost similar if an invertible operator $\mathcal{N}$ exists such that

$A^*A = \mathcal{N}^{-1}(B^*B)\mathcal{N}$ and $A^* + A = \mathcal{N}^{-1}(B^* + B)\mathcal{N}$. This property on operators has been studied by other scholars and variety of results given which demonstrated metric equivalence and almost similarity to be equivalence relations. Also, almost similar operators have been found to preserve nullity and co-nullity of operators and almost similarity implies similarity.

1.2 Basic Concepts

The following are terms which were employed in our research:

Definition 1.2.1. A *Hilbert space* is a complete normed linear space with the structure of inner product defined on it.

Definition 1.2.2. *An operator* is a mapping that acts on elements of a domain space to generate images of another space (relatively the same space).

Definition 1.2.3. An operator $\mathcal{T} \in \mathfrak{B}(H)$ is *bounded* or enclosed if there exists $\mathcal{C} > 0$ such that,

$\|\mathcal{T}x\| \leq \mathcal{C}\,\|x\|, \forall\, x \in H$ where, $\|\mathcal{T}\| = \inf\{\mathcal{C} > 0 : \|\mathcal{T}x\| \leq \mathcal{C}\,\|x\|, \forall\, x \in H\}$.

Definition 1.2.4. An operator $\mathcal{T} \in \mathfrak{B}(H)$ is an *isometry* if $\mathcal{T}^*\mathcal{T} = I$, where I is the identity operator and it is *partial isometry* if $\mathcal{T}\mathcal{T}^*\mathcal{T} = \mathcal{T}$ or equivalently, if $\mathcal{T}^*\mathcal{T}$ is a projection. Thus, it is a distance-preserving transformation. Thus, it is a linear transformation which is isometric on the orthogonal complement of its kernel.

Definition 1.2.5. Let $\mathcal{L}: \mathcal{V} \longrightarrow \mathcal{W}$, where $\mathcal{V}$ and $\mathcal{W}$ are vector spaces, be a linear map. Then the set of all vectors v for which $\mathcal{L}(v) = 0$, with 0 denoting the zero vector in $\mathcal{W}$, is the *kernel of* $\mathcal{L}$. That is, $Ker\,(\mathcal{L}) = \{v \in \mathcal{V} : \mathcal{L}(v) = 0\}$ .Thus, kernel is simply a null space.

Definition 1.2.6. For an operator $A \in \mathfrak{B}(H)$, its *nullity*, denoted by $nul(A)$, is defined as $nul(A) = \dim(\ker(A))$ and its *co-nullity* is the dimension of $\ker(A^*)$.

Definition 1.2.7. Let $A \in \mathcal{B}(H_1, H_2)$. Then A is said to be an *invertible operator* if there exists an operator $A^{-1} \in \mathcal{B}(H_2, H_1)$ such that, $A^{-1}Ax = x$ and $AA^{-1}y = y$ for every $x \in H_1$ and for every $y \in H_2$ respectively. Then the operator A^{-1} is referred to as the inverse of A.

Definition 1.2.8. Let $A \in \mathfrak{B}(H)$. Then A is said to be *positive normal* or simply *posinormal* if there exist a positive operator $P \in \mathfrak{B}(H)$ such that, $AA^* = A^*PA$. If A^* is posinormal, then A is said to be *coposinormal.*

Definition 1.2.9 Let $\mathcal{T} \in \mathfrak{B}(H)$. The *posispectrum* of $\mathcal{T}$, denoted by P($\mathcal{T}$), is the set $\{\lambda : \mathcal{T} - \lambda$ is not posinormal$\}$.

Definition 1.2.10. A bounded linear operator $A \in \mathfrak{B}(H)$ is said to be *hyponormal* if $AA^* \leqq A^*A$ and *quasinormal* if A commutes with A^*A.

Definition 1.2.11. An operator $A \in \mathfrak{B}(H)$ is *heminormal* if A is hyponormal and A^*A commutes with AA^*.

Definition 1.2.12. An operator $\mathcal{T} \in \mathfrak{B}(H)$ for which $\mathcal{T}^*\mathcal{T}$ and $\mathcal{T} + \mathcal{T}^*$ commute is called a θ*-operator.* The class of θ-operators is denoted by θ.

Definition 1.2.13. Let $\mathcal{T} \in \mathfrak{B}(H)$. The *spectrum* of $\mathcal{T}$, denoted by $\sigma(\mathcal{T})$ is the set of all $\lambda \in \mathbb{C}$, such that the operators $\mathcal{T} - \lambda I$ is not invertible. In finite-dimensional spaces $\sigma(\mathcal{T})$, consists of the eigenvalues of $\mathcal{T}$.

Definition 1.2.14. The following are fundamental *classes of operators* which are significant to this study. An operator $\mathcal{T} \in \mathfrak{B}(H)$ is said to be;

- *Hermitian* or *self-adjoint* if $\mathcal{T}^* = \mathcal{T}$,
- *Normal* if $\mathcal{T}^*\mathcal{T} = \mathcal{T}\mathcal{T}^*$,
- *Unitary* if $\mathcal{T}^*\mathcal{T} = \mathcal{T}\mathcal{T}^* = I$,
- *Skew adjoint* if $\mathcal{T}^* = -\mathcal{T}$,
- *Binormal* if $(\mathcal{T}^*\mathcal{T})(\mathcal{T}\mathcal{T}^*) = (\mathcal{T}\mathcal{T}^*)(\mathcal{T}^*\mathcal{T})$,
- *Orthogonal projection* if $\mathcal{T}^2 = \mathcal{T}$ and $\mathcal{T}^* = \mathcal{T}$,
- *Isometric* if $\mathcal{T}^*\mathcal{T} = I$,

- *Partial isometry* if $\mathcal{T}\mathcal{T}^*\mathcal{T} = \mathcal{T}$,
- *A Self-adjoint* if $\mathcal{T}^* = A^{-1}\mathcal{T}A$, for some operator A which is self-adjoint and invertible.
- *Normaloid* if $r(\mathcal{T}) = \|\mathcal{T}\|$ or $\|\mathcal{T}^n\| = \|\mathcal{T}\|^n$.
- *Cohyponormal* if $\mathcal{T}\mathcal{T}^* \geq \mathcal{T}^*\mathcal{T}$ i.e its adjoint is hyponormal
- *Subnormal* if there exists a Hilbert Space K containing H and a normal operator $\mathcal{N}$ acting on K such that H is $\mathcal{N}$-invariant and $\mathcal{T}$ is the restriction of $\mathcal{N}$ onto H. Thus, $\mathcal{T} \in \mathfrak{B}(H)$ is said to be a subnormal if H is a subspace of a Hilbert space K and with respect to the decomposition $K = H \oplus H^{\perp}$, the operator N has the triangular block representation given by $N = \begin{pmatrix} \mathcal{T} & B \\ 0 & C \end{pmatrix}$, where $B: H^{\perp} \longrightarrow H$ and $C: H^{\perp} \longrightarrow H^{\perp}$.
- *M-hyponormal* if there exist a real number $M \geq 0$ such that $\|(\mathcal{T} - zI)^* x\| \leq M\|(\mathcal{T} - zI)\|$ $\forall\, x \in H$ and for every complex number z.
- *Dominant* if for any complex number λ, there exists a number $M_\lambda \geq 1$ such that $(\mathcal{T} - \lambda I)(\mathcal{T} - \lambda I)^* \leq M_\lambda^2(\mathcal{T} - \lambda I)^*(\mathcal{T} - \lambda I)$. Equivalently $Ran(\mathcal{T} - \lambda I) \subset Ran(\mathcal{T} - \lambda I)^*, \forall\ \lambda \in \sigma(\mathcal{T})$.
- *Semi normal* if either $\mathcal{T}$ or $\mathcal{T}^*$ is hyponormal.

Hence we have the following class inclusions;

(i) {Unitary operators} $\subseteq$ {isometric operators} $\subseteq$ {partial isometries}.

{Unitary operators} $\subseteq$ {co-isometric operators} $\subseteq$ {partial isometries}.

(ii) {Normal operators} $\subseteq${Quasinormal operators} $\subseteq$ {Binormal operators}

(iii) {Positive operators}$\subseteq${Self-adjoint operators}$\subseteq${Normal operators}$\subseteq$

{ θ-operators}.

(iv) {projection}⊆{positive}⊆{self-adjoint}⊆{Normal}⊆{Hyponormal operators}

⊆{M-hyponormal}⊆{Dominant operators} ⊆ {Posinormal operators}.

Definition 1.2.15. Two operators $A, B \in \mathfrak{B}(H)$ are *similar*, denoted by $A \sim B$, if there exists an invertible operator X such that $XA = BX$ or equivalent $A = X^{-1}BX$.

Similarly, two operators $A, B \in \mathfrak{B}(H)$ are *unitarily equivalent* and denoted by $A \cong B$ if there is a unitary operator U such that $UA = BU$ i.e, $A = U^*BU$ or $B = UAU^*$.

Computation shows that similarity is a equivalence relation in $\mathfrak{B}(H)$ and also that similar operators in $\mathfrak{B}(H)$ have the same:

i. *Spectrum*, denoted by $\sigma(\mathcal{T}) = \{\lambda \in \mathbb{C}: (\mathcal{T} - \lambda I) \text{ is not invertible}\}$.

   The complement of the spectrum of $\mathcal{T}$ is called the resolvent set of $\mathcal{T}$.

ii. *Point spectrum*, denoted by

   $\sigma_p(\mathcal{T}) = \{\lambda \in \mathbb{C}: Ker(\mathcal{T} - \lambda I) \neq (0)\}\ i.e\ \mathcal{T}x = \lambda x\ , for\ x \in H.$

iii. *Approximate point spectrum*, denoted by $\sigma_{ap}(\mathcal{T})$, if there exists a sequence of unit vector $\{x_n\}$ such that, $\|(\mathcal{T} - \lambda I)x_n\| \longrightarrow 0$.

Definition 1.2.16. An operator $\mathcal{T} \in \mathfrak{B}(H)$ is referred to as a *quasiaffinity* if $\mathcal{T}$ is both one-one and has dense range. Quasiaffinity is also referred to as a *quasi-invertible*.

Definition 1.2.17. Two operators $A \in \mathfrak{B}(H)$ and $B \in \mathfrak{B}(K)$ are *quasisimilar* if there are quasi-invertible operators X from K to H and Y from H to K which satisfy the equations; $XA = BX$ and $BY = YA$. In all classes of operators, *quasisimilarity* is an equivalence relation.

Definition 1.2.18. Let two operators $A, B \in \mathfrak{B}(H)$. Then A and B are *almost similar* and denoted as $A \overset{a.s}{\approx} B$, if an invertible operator $\mathcal{N}$ exists such that $A^*A = \mathcal{N}^{-1}(B^*B)\mathcal{N}$ and $A^* + A = \mathcal{N}^{-1}(B^* + B)\mathcal{N}$.

1.3. Statement of the problem

A number of authors have studied the concepts of similarity and quasisimilarity of operators in Hilbert spaces. In the recent past, studies have also been done on the concept of almost similarity of operators, unitarily equivalent operators and metrically equivalent operators which established some relationships between them. Almost similarity has also been shown to be an equivalence relation. However, there is little known in literature on properties of almost similarity on partial isometries, θ-operators and posinormal operators. The conditions under which almost similarity give equality of spectra had not been established. This research therefore sought to investigate almost similarity property on partial isometries, θ-operators, posinormal operators and conditions under which almost similarity gives equality of spectra.

1.4. Objectives of the Study

1.4.1 Main objective

The main objective of this research was to extend the study on similarity and quasisimilarity to that of almost similarity of operators in Hilbert spaces.

1.4.2 Specific objectives

The specific objectives of this research were as follows;

i. To investigate properties of almost similarity on partial isometries.

ii. To investigate properties of almost similarity on θ-operators.

iii. To investigate properties of almost similarity on posinormal operators.

iv. To look for conditions under which almost similarity property yields equal spectra for such operators.

1.5 Significance of the study

The understanding of almost similarity properties on the partial isometries, θ-operators and establishing conditions in which this property yields equality of spectrum will be significant in knowledge value addition on interpretation of operator relations on Hilbert spaces. Operators form the most intrinsic part of the formulation of theories in quantum mechanics, spectral analysis of functions including theories of wavelength, partial differential equations such as formulation of the Dirichlet problems and unitary group representations. Also, studies on Hilbert spaces not only clarify but also generalize the concept of Fourier expansion and certain linear transformations such as the Fourier transform.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Since the conception of Hilbert spaces, significant studies have been done to establish structural composition and relations under different operators in this space as highlighted below;

Hoover (1972) researched hyperinvarint subspaces and proved the result that, for two operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ such that, if $\mathcal{S}$ and $\mathcal{T}$ are quasisimilar operators acting on Hilbert spaces H and K respectively, then $\mathcal{T}$ has a hyperinvariant subspace whenever $\mathcal{S}$ has. Also, if $\mathcal{S}$ is normal, then the lattice of hyperinvariant substances for $\mathcal{T}$ contains a sub-lattice which is lattice isomorphic to the lattice of spectral projection for $\mathcal{S}$.

Hoover further showed some properties of operators that are preserved by quasisimilarity and those that are not. He also showed that quasisimilar normal operators are unitarily equivalent, quasisimilar isometries are unitary equivalent and finally quasisimilarity doesn't preserve spectra and compactness.

Kubrusty (1997) studied hyperinvariance and demonstrated that similarity preserves nontrivial subspaces while quasisimilarity preserves hyperinvariant subspaces.

Douglas (1969) proved the equality of spectra of quasisimilar normal operators. This was then extended to two pairs of quasisimilar, hypernormal operators by Halmos (1976).

Clary (1975) showed that quasisimilar-hypernormal operators have equal spectra.

Stampfli (1981) proved the result on quasisimilar hyponormal operators to be an inclusion for the spectra of quasisimilar operators satisfying Dunfords condition.

Lambert (2001) extensively showed that any two unilateral weighted shifts which are Quasisimilar are similar but if unilateral shift is bilateral as by L.A Fialkow the result failed.

Williams (1980) showed that quasisimilar-hyponormal operators have equal spectra and established results relating hyponormal and quasisimilarity operators. He further justified that if one of the quasiaffinities of two hyponormal operators is compact, then they have equal essential spectra.

Lee (1995) extended Williams result of equality of essential spectra of certain quasisimilar-quasihyponormal operators and also demonstrated that quasisimilarity preserves the Fredholm property.

Jeon and Duggal (2004) showed that normal part of quasisimilar p-hyponormal operators are unitarily equivalent and also that a p-hyponormal spectral operator is normal. They also extended similar results to quasisimilar injective. P-quasihyponormal operators were also shown to have the same spectra and essential spectra.

The property of almost similarity on classes of operators was first introduced by Jibril (1996).

Nzimbi *et al.* (2008) proved that the property of almost similarity is an equivalence relation. They further studied the concept of almost similarity and established that similarity implies almost similarity under certain conditions.

Nzimbi *et al.* (2013) introduced the concept of metric equivalence and proved that metric equivalence is also an equivalence relation. They also discussed the spectral picture of metrically equivalent operators and further gave conditions under which the metric equivalence of operators implies unitary equivalence of operators.

Nzimbi *et al.* (2016) showed that almost similarity and metric equivalence preserve nullity and co-nullity of operators. They further showed that quasisimilar and quasiaffine normal operators have equal spectra.

Kipkemboi (2016) studied almost similarity and other related equivalence relations of operators in Hilbert Spaces. He also showed that two orthogonal projections acting on a Hilbert space are Murray-von Neumann equivalent. Further, he showed almost similar operators are equivalent if any only if there exists a partial isometry acting on the two operators. He also proved that unitary equivalent operators are stably unitarily equivalent.

In reference to all these results, it is evident that there are gaps on properties of almost similarity on partial isometries, θ-operators, posinormal operators among others, and also conditions under which almost similarity gives equality of spectra. This research is intended to attempt the above problem and make appropriate conclusions based on the results obtained at the end.

2.2. Research Methodology

The knowledge of properties of operators in Hilbert space including adjoints, spectrum, invariant subspaces, sesquilinear maps, quadratic forms, projections and θ-operators was used in this study. Also, the basic properties of Functional Analysis on norms, the spectral radius formula, open mapping theorem, the uniform boundedness principle and Riesz representation theorem are core foundation for this research. Finally, basic Measure Theory on bounded convergence theorem and Von-Neumann Schatten ideals were utilized as well in comparing, analyzing and making conclusions in this research.

CHAPTER THREE

ALMOST SIMILARITY PROPERTY ON PARTIAL ISOMETRIES

3.1 Introduction

In this chapter we determined whether two almost similar operators belong to the same class with consideration to the class of partial isometries together with its subclasses which are as follows; {Unitary} $\subseteq$ {Isometry} $\subseteq$ {Partial Isometry} and {Unitary} $\subseteq$ {co-isometry} $\subseteq$ {Partial isometry}.The question of whether such operators have equal spectra were considered.

However, we first investigated the general properties and unitary equivalence for such operators.

3.2. General properties of partial isometries

The following results had been established in this area:

Lemma 3.2.1(Skoufranis, 2014). If $\mathcal{T} \in \mathfrak{B}(H)$ is an isometry, then

$\langle \mathcal{T}x, \mathcal{T}y\rangle = \langle x, y\rangle, \forall\, x, y \in H$.

Proof: Given that $x, y \in H$, then it follows that;

$$\begin{aligned}\|x\|^2 + 2Re(\langle x, y\rangle) + \|y\|^2 &= \langle\, x+y, x+y\rangle \\ &= \|\, x+y\|^2 \\ &= \|\mathcal{T}(x+y)\|^2 \\ &= \langle\, \mathcal{T}(x+y), \mathcal{T}(x+y)\rangle \\ &= \|\mathcal{T}x\|^2 + 2Re(\langle \mathcal{T}x, \mathcal{T}y\rangle) + \|\mathcal{T}y\|^2 \\ &= \|x\|^2 + 2Re(\langle \mathcal{T}x, \mathcal{T}y\rangle) + \|y\|^2.\end{aligned}$$

Hence, $Re(\langle x, y\rangle) = Re(\langle \mathcal{T}x, \mathcal{T}y\rangle)$.

By replacing $\mathcal{y}$ with $i\mathcal{y}$ and repeating the process above, it yields;

$$\begin{aligned} I_m(\langle x, \mathcal{y}\rangle) &= Re(-i\langle x, \mathcal{y}\rangle) \\ &= Re(-i\langle \mathcal{T}x, \mathcal{T}\mathcal{y}\rangle) \\ &= I_m(\langle \mathcal{T}x, \mathcal{T}\mathcal{y}\rangle). \end{aligned}$$

We therefore conclude that $\langle \mathcal{T}x, \mathcal{T}\mathcal{y}\rangle = \langle x, \mathcal{y}\rangle$ as desired. ■

Proposition 3.2.2 (Nzimbi *et al*., 2008). An operator $\mathcal{T} \in \mathfrak{B}(H)$ is an isometry if and only if $\mathcal{T}^*\mathcal{T} = I$.

Proof: If $\mathcal{T} \in \mathfrak{B}(H)$ is an isometry, then for $x, \mathcal{y} \in H$, we have;

$\langle \mathcal{T}x, \mathcal{T}y\rangle = \langle x, \mathcal{y}\rangle \; \forall \; x, \mathcal{y} \in H$. Thus,

$\langle (I - \mathcal{T}^*\mathcal{T})x, \mathcal{y}\rangle = 0, \forall \; x, \mathcal{y} \in H.$

Hence, $\mathcal{T}^*\mathcal{T} = I$. Also, for $x \in H$, then we have $\|\mathcal{T}x\|^2 = \langle \mathcal{T}x, \mathcal{T}x\rangle$

$$\begin{aligned} &= \langle \mathcal{T}^*\mathcal{T}\, x, x\rangle \\ &= \langle x, x\rangle \\ &= \|x\|^2 \end{aligned}$$

Hence, $\|\mathcal{T}x\| = \|x\|, \; \forall \; x \in H$. This shows that $\mathcal{T}$ is an isometry. ■

From this proposition, we now consider the general properties of partial isometries in the next theorem.

Theorem 3.2.3. (Salhi and Zerovali, 2019)**.** For $\mathcal{T} \in \mathfrak{B}(H)$, the following statements are equivalent;

(i) $\mathcal{T}$ is a partial isometry,

(ii) $\mathcal{T}^*$ is a partial isometry,

(iii) $\mathcal{T}\mathcal{T}^*$ is a projection,

(iv) $\mathcal{T}^*\mathcal{T}$ is a projection,

(v) $\mathcal{T}^*\mathcal{T}\,\mathcal{T}^* = \mathcal{T}^*$

(vi) $\mathcal{T}\mathcal{T}^*\mathcal{T} = \mathcal{T}$.

The range of $\mathcal{T}$ is closed, $\mathcal{T}\mathcal{T}^*$ is a projection onto $ran\,(\mathcal{T})$ and $\mathcal{T}^*\mathcal{T}$ is the projection onto $ker(\mathcal{T})^{\perp}$

Proof: If $\mathcal{T}$ is a partial isometry, (i) $\Longrightarrow$ (v). For an element $x \in H$ and considering $\langle \mathcal{T}^*\mathcal{T}\mathcal{T}^*x, y\rangle$ and $\langle \mathcal{T}^*x, y\rangle$, for $y \in H$. Therefore, if $y \in ker\,(\mathcal{T})$, then

$$\begin{aligned}\langle \mathcal{T}^*\mathcal{T}\mathcal{T}^*x, y\rangle &= \langle \mathcal{T}^*\mathcal{T}x, \mathcal{T}y\rangle \\ &= 0 \\ &= \langle x, \mathcal{T}y\rangle\end{aligned}$$

Since $\mathcal{T}$ is an isometry then $\ker\,(\mathcal{T})^{\perp} = ran\,(\mathcal{T}^*)$. Thus, the inner product of two elements of $ran\,(\mathcal{T}^*)$ is shown.

Supposing $y \in \ker(\mathcal{T})^{\perp} = ran(\mathcal{T}^*)$, then, $\langle \mathcal{T}^*\mathcal{T}\mathcal{T}^*x, y\rangle = \langle \mathcal{T}(\mathcal{T}^*x), \mathcal{T}y\rangle$

$$= \langle \mathcal{T}^*x, y\rangle.$$

For $\ker(\mathcal{T}) \oplus \ker(\mathcal{T})^{\perp} = H$, it follows that $\langle \mathcal{T}^*\mathcal{T}\mathcal{T}^*x, y\rangle = \langle \mathcal{T}^*x, y\rangle\ \forall\ x, y \in H$.

Hence, $\mathcal{T}^*\mathcal{T}\mathcal{T}^* = \mathcal{T}^*$ if and only if $\mathcal{T} = (\mathcal{T}^*)^* = (\mathcal{T}^*\mathcal{T}\mathcal{T}^*)^* = \mathcal{T}\mathcal{T}^*\mathcal{T}$.

Thus, (v) $\Longrightarrow$ (vi).

If $\mathcal{T}\mathcal{T}^*$ is self-adjoint, then $\mathcal{T}\mathcal{T}^*\mathcal{T}\mathcal{T}^* = \mathcal{T}(\mathcal{T}^*\mathcal{T}\mathcal{T}^*)$

$$= \mathcal{T}\mathcal{T}^*.$$

Therefore, (iii) and (iv) are also consequence of (v). Hence, $\mathcal{T}\,\mathcal{T}^*$ is a projection.

Equivalently, if $\mathcal{T}\mathcal{T}^*$ is self-adjoint, then we have

$$\begin{aligned}\mathcal{T}^*\mathcal{T}\mathcal{T}^*\mathcal{T} &= (\mathcal{T}^*\mathcal{T}\mathcal{T}^*)\mathcal{T} \\ &= \mathcal{T}^*\mathcal{T}.\end{aligned}$$

Thus, $\mathcal{T}^*\mathcal{T}$ is also a projection.

Finally, taking $\ker(\mathcal{T})^{\perp} = \overline{ran(\mathcal{T}^*)}$, then for a projection $\mathcal{T}\mathcal{T}^*$, there exists a sequence $(x_y)_{y \geq 1} \in H$, such that, $lim_{y \to \infty} \mathcal{T}^* x_y = x$. Therefore,

$$\begin{aligned}
\|\mathcal{T}x\|^2 &= lim_{y \to \infty} \|\mathcal{T}\mathcal{T}^* xy\|^2 \\
&= lim_{y \to \infty} \langle \mathcal{T}\mathcal{T}^* xy, \mathcal{T}\mathcal{T}^* xy \rangle \\
&= lim_{y \to \infty} \langle (\mathcal{T}\mathcal{T}^*)^2 xy, xy \rangle \\
&= lim_{y \to \infty} \langle \mathcal{T}\mathcal{T}^* xy, xy \rangle \\
&= lim_{y \to \infty} \|\mathcal{T}\mathcal{T}^* xy\|^2 \\
&= \|x\|^2.
\end{aligned}$$

Thus, $\mathcal{T}$ is a partial isometry, $\|\mathcal{T}x\| = \|x\|, \forall\, x \in \ker(\mathcal{T})^{\perp}$. ■

We then have the following remark.

Remark 3.2.4. For an operator $\mathcal{T} \in \mathfrak{B}(H)$ such that, $\mathcal{T}\mathcal{T}^*$ is a projection, we have that the adjoint of $\mathcal{T}$ is a partial isometry.

3.3 Unitary equivalence of subclasses of partial isometries

The following results outline the general properties of unitary equivalence on subclasses of partial isometries.

Theorem 3.3.1. (Luketero and Khalagai, 2020). Let $\mathcal{S} \in \mathfrak{B}(H)$ be a partial isometry and $\mathcal{T}$ be any other operator such that either,

(i) $\mathcal{S} = U\mathcal{T}U^*$ where U is an isometry or

(ii) $\mathcal{S} = U^*\mathcal{T}U$ where U is a co-isometry.

Then, $\mathcal{T}$ is also a partial isometry.

Proof: (i) If $\mathcal{S} = U\mathcal{T}U^*$ then $\mathcal{S}^* = U\mathcal{T}^*U^*$. Since $\mathcal{S}$ is a partial isometry, then we have $\mathcal{S} = \mathcal{S}\mathcal{S}^*\mathcal{S}$. Therefore,

$$\mathcal{S} = \mathcal{S}\mathcal{S}^*\mathcal{S}$$
$$= U\mathcal{T}U^*U\mathcal{T}^*U^*U\mathcal{T}U^*$$
$$= U\mathcal{T}\mathcal{T}^*\mathcal{T}U^*$$
$$= U\mathcal{T}U^*$$

Thus,

$$U\mathcal{T}\mathcal{T}^*\mathcal{T}U^* = U\mathcal{T}U^* \tag{1}$$

Now pre-multiplying by U^* and post-multiplying by U in (1) above, we obtain;

$\mathcal{T}\mathcal{T}^*\mathcal{T} = \mathcal{T}$. Therefore $\mathcal{T}$ is also a partial isometry.

(ii) Again, if $\mathcal{S} = U^*\mathcal{T}U$ then $\mathcal{S}^* = U^*\mathcal{T}^*U$.

Since $\mathcal{S}$ is a partial isometry, we have that $\mathcal{S} = \mathcal{S}\mathcal{S}^*\mathcal{S}$. It therefore, follows that

$$\mathcal{S} = \mathcal{S}\mathcal{S}^*\mathcal{S}$$
$$= U^*\mathcal{T}UU^*\mathcal{T}^*UU^*\mathcal{T}U$$
$$= U^*\mathcal{T}\mathcal{T}^*\mathcal{T}U$$
$$= U^*\mathcal{T}U$$

Thus,

$$U^*\mathcal{T}\mathcal{T}^*\mathcal{T}U = U^*\mathcal{T}U \tag{2}$$

Pre-multiplying by U and post-multiplying by U^* in (2) above, $\mathcal{T}\mathcal{T}^*\mathcal{T} = \mathcal{T}$ is obtained.

Therefore, $\mathcal{T}$ is also a partial isometry. ■

Corollary 3.3.2. Let $\mathcal{S} \in \mathfrak{B}(H)$ be a partial isometry and $\mathcal{T}$ be any other operator such that either $\mathcal{S} = U\mathcal{T}U^*$ or $\mathcal{S} = U^*\mathcal{T}U$ where U is unitary. Then $\mathcal{T}$ is also a partial isometry.

Proposition 3.3.3. (Nzimbi *et al.*, 2016). An operator $\mathcal{S} \in \mathfrak{B}(H)$ is quasi-unitary if and only if $(I - \mathcal{S})$ is unitary.

Proof: If $\mathcal{S}$ is quasi-unitary, then $(I-\mathcal{S})^*(I-\mathcal{S}) = (I-\mathcal{S})(I-\mathcal{S})^* = I$.

Hence, $I-\mathcal{S}$ is unitary.

Conversely, suppose $I-\mathcal{S}$ is unitary, then we have;

$I-(\mathcal{S}+\mathcal{S}^*)+\mathcal{S}^*\mathcal{S} = I-(\mathcal{S}^*+\mathcal{S})+\mathcal{S}\mathcal{S}^* = I$. Thus, $\mathcal{S}^*\mathcal{S} = \mathcal{S}\mathcal{S}^* = \mathcal{S}+\mathcal{S}^*$. If

$V = I-\mathcal{S}$, then $V^*V = VV^* = I$.This implies that it is unitary.

Again, suppose $I-\mathcal{S}$ is unitary, i.e. $(I-\mathcal{S})^*(I-\mathcal{S}) = (I-\mathcal{S})(I-\mathcal{S})^* = I$, which can be simplified as follows; $I-\mathcal{S}-\mathcal{S}^*+\mathcal{S}^*\mathcal{S} = I-\mathcal{S}-\mathcal{S}^*+\mathcal{S}\mathcal{S}^* = I$

$$-(\mathcal{S}+\mathcal{S}^*)+\mathcal{S}^*\mathcal{S} = -(\mathcal{S}+\mathcal{S}^*)+\mathcal{S}\mathcal{S}^* = 0.$$

Then, $\mathcal{S}^*\mathcal{S} = \mathcal{S}+\mathcal{S}^*$ and $\mathcal{S}^*\mathcal{S} = \mathcal{S}\mathcal{S}^* = \mathcal{S}+\mathcal{S}^*$. Hence $\mathcal{S}$ is quasi-unitary. ∎

3.4 Almost similarity of subclasses of partial isometries.

Proposition 3.4.1. (Nzimbi *et al*., 2008). Unitarily equivalent operators are almost similar.

Proof: Let $\mathcal{S},\mathcal{T} \in \mathfrak{B}(H)$ be unitarily equivalent. Then there exist a unitary operator U such that

$\mathcal{S} = U^*\mathcal{T}U$. This implies that, $\mathcal{S}^* = U^*\mathcal{T}^*U$.

Thus, $\mathcal{S}^*\mathcal{S} = U^*\mathcal{T}^*UU^*\mathcal{T}U$

$= U^{-1}\mathcal{T}^*\mathcal{T}U$ and

$\mathcal{S}^*+\mathcal{S} = U^*\mathcal{T}^*U + U^*\mathcal{T}U$

$= U^*(\mathcal{T}^*+\mathcal{T})U$

$= U^{-1}(\mathcal{T}^*+\mathcal{T})U.$

Hence, $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$. ∎

Proposition 3.4.2. (Jibril, 1996a). Let $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$. If $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ and $\mathcal{T}$ is isometric, then $\mathcal{S}$ is isometric.

Proof: Since $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ then $\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})\mathcal{N}$ and $\mathcal{S}^* + \mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\mathcal{N}$. Since $\mathcal{T}$ is an isometry, i.e, $\mathcal{T}^*\mathcal{T} = I$, it follows that $\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}(I)\mathcal{N}$. Hence, $\mathcal{S}$ is isometric. ■

Proposition 3.4.3. (Jibril, 1996a). An operator $\mathcal{T} \in \mathfrak{B}(H)$ is isometric if and only if, $\mathcal{T} \overset{a.s}{\sim} U$, for some unitary operator U.

Proof: Supposing that $\mathcal{T} \overset{a.s}{\sim} U$, for some unitary operator U then an invertible operator $\mathcal{N}$ exists such that, $\mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})\mathcal{N} = U^*U = I$. This implies that, $\mathcal{T}^*\mathcal{T} = \mathcal{T}\mathcal{T}^* = I$.

Hence, $\mathcal{T}$ is isometric. From $\mathcal{T}^*\mathcal{T} = \mathcal{T}\mathcal{T}^* = I$, it follows that $\mathcal{T}^*\mathcal{T} = \mathcal{T}^{-1}(\mathcal{T}^*)^*\mathcal{T}^*I$ and $\mathcal{T}^* + \mathcal{T} = \mathcal{T} + \mathcal{T}^*$. Thus, $\mathcal{T}^* + \mathcal{T} = I^{-1}[(\mathcal{T}^*)^*\mathcal{T}^*]\,\mathcal{T}$. Hence, $\mathcal{T} \overset{a.s}{\sim} \mathcal{T}^* = \mathcal{T}^{-1}$.

However, if $\mathcal{T} \in \mathfrak{B}(H)$ is such that, $\mathcal{T} \overset{a.s}{\sim} \mathcal{T}^{-1}$, then $\mathcal{T}$ is not necessarily unitary. ■

Remark 3.4.4. The following corollary is immediate from the Proposition 3.4.4 above.

Corollary 3.4.5. (Nzimbi *et al*., 2008)**.** An isometry which is almost similar to a unitary operator is itself unitary.

Proposition 3.4.6. (Jibril, 1996a). If $\mathcal{T} \in \mathfrak{B}(H)$ is invertible, such that $\mathcal{T} \overset{a.s}{\sim} U$, for some unitary operator, $U \in \mathfrak{B}(H)$ then $\mathcal{T}$ is unitary.

Proof: Given that $\mathcal{T} \overset{a.s}{\sim} U$, then there exist an invertible operator $\mathcal{N}$ such that

$\mathcal{T}^*\mathcal{T} = \mathcal{N}^{-1}U^*U\mathcal{N} = I$. Therefore, $\mathcal{T}^{*-1}\mathcal{T}^*\mathcal{T}\mathcal{T}^{-1} = \mathcal{T}^{*-1}\mathcal{T}^{-1}$.

Since, $\mathcal{T}^{*-1}\mathcal{T}^*\mathcal{T}\mathcal{T}^{-1} = I$, we have $\mathcal{T}^{*-1}\mathcal{T}^{-1} = (\mathcal{T}\mathcal{T}^*)^{-1}$and thus $\mathcal{T}\mathcal{T}^* = I$. Hence,

$\mathcal{T}^*\mathcal{T} = \mathcal{T}\mathcal{T}^* = I$. This shows that $\mathcal{T}$ is unitary. ■

Proposition 3.4.7. (Jibril, 1996b). If two operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ are such that, $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$, and $\mathcal{S}$ is a partially isometric, then so is $\mathcal{T}$.

Proof: For $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$, then an invertible operator $\mathcal{N}$ exists such that $\mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})\mathcal{N} = \mathcal{S}^*\mathcal{S}$. Since $\mathcal{S}$ is partially isometric, i.e. $(\mathcal{S}^*\mathcal{S})^2 = \mathcal{S}^*\mathcal{S}$, it implies that,

$$[\mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})\mathcal{N}][\mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})\mathcal{N}] = \mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})\mathcal{N}.$$

$$\text{Thus, } \mathcal{N}^{-1}\mathcal{T}^*\mathcal{T}\mathcal{T}^*\mathcal{T}\mathcal{N} = \mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})\mathcal{N}.$$

It follows that $(\mathcal{T}^*\mathcal{T})^2 = \mathcal{T}^*\mathcal{T}$. Hence, $\mathcal{T}^*\mathcal{T}$ is a projection, justifying that, $\mathcal{T}$ is partially isometric. ■

Theorem 3.4.8. Let $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ such that $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$. If $\mathcal{S}^2$ is a partial isometry and $\mathcal{T}$ is self-adjoint, then $\mathcal{T}^2$ is also partially isometric.

Proof: Since $\mathcal{S}^2$ is a partial isometry, we have $\mathcal{S}^2 = \mathcal{S}^2\mathcal{S}^{*2}\mathcal{S}^2$ and by projection property, we also have $\mathcal{S}\mathcal{S}^* = \mathcal{S}\mathcal{S} = \mathcal{S}^2$.

$\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$, implies there exists an invertible operator $\mathcal{N}$ such that

$$\mathcal{T}^*\mathcal{T} = \mathcal{N}^{-1}\mathcal{S}^*\mathcal{S}\mathcal{N} \tag{3}$$

and

$$\mathcal{T}^* + \mathcal{T} = \mathcal{N}^{-1}(\mathcal{S}^* + \mathcal{S})\mathcal{N} \tag{4}$$

From $\mathcal{S}\mathcal{S}^* = \mathcal{S}\mathcal{S} = \mathcal{S}^2$, then (3) becomes,

$$\mathcal{T}^*\mathcal{T} = \mathcal{N}^{-1}\mathcal{S}^2\mathcal{N}.$$

Consequently, $\mathcal{T}^*\mathcal{T} = \mathcal{T}^2$ and thus, $\mathcal{T}^2 = \mathcal{N}^{-1}\mathcal{S}^2\mathcal{N}$. It follows that, $\mathcal{T}^2 = \mathcal{T}^2\mathcal{T}^{*2}\mathcal{T}^2$, which is equivalent to $\mathcal{T}^2 - \mathcal{T}^2\mathcal{T}^{*2}\mathcal{T}^2 = 0$

$$\mathcal{T}^2(1 - \mathcal{T}^{*2}\mathcal{T}^2) = 0.$$

Implying that $\mathcal{T}^{*2}\mathcal{T}^2 = 1,$ or $\mathcal{T}^*\mathcal{T} = 1$

Using (4), it follows that, $(\mathcal{T}^* + \mathcal{T})^2 = \mathcal{T}^{*2}+2\mathcal{T}^2 + \mathcal{T}^2 = 4\mathcal{T}^2$.

Hence, $\mathcal{T}^2$ is a partial isometry as claimed. ■

CHAPTER FOUR

ALMOST SIMILARITY PROPERTY ON θ-OPERATORS

4.1 Introduction.

In this chapter, almost similarity property on θ-operators is investigated. Further, we study α- almost similarity concept on these θ-operators. The subclasses are as follows: {Projection operators}⊆{Positive operators}⊆{Self-adjoint operators}⊆{Normal operators}⊆{θ-operators}.

The following theorems explain the significant basic properties of θ-operators.

4.2. General properties of θ-Operators

The following entails results on general properties of θ-operators;

Recall, an operator $\mathcal{T} \in \mathfrak{B}(H)$ is referred to as a $\theta-$operator, if $\mathcal{T}^*\mathcal{T}$ commutes with $\mathcal{T}^* + \mathcal{T}$.

For an operator $\mathcal{S} \in \theta$, $4\mathcal{S}^*\mathcal{S} - (\mathcal{S}^* + \mathcal{S})^2 \geq 0$.

Define $\mathcal{T} = \mathcal{S}^* + \mathcal{S} + i\sqrt{4\mathcal{S}^*\mathcal{S} - (\mathcal{S}^* + \mathcal{S})^2/2}$. Then $\mathcal{T}$ is normal and $\sigma(\mathcal{S})$ is contained in the closed upper half plane. It follows that

$\mathcal{T}^*\mathcal{T} = \mathcal{S}^*\mathcal{S}$ and $\mathcal{T}^* + \mathcal{T} = \mathcal{S}^* + \mathcal{S}$.

Therefore, for an operator $\mathcal{S} \in \mathfrak{B}(H)$, if $\mathcal{S}$ is self-adjoint, then $\mathcal{S}^*\mathcal{S}$ commute with $\mathcal{S}^* + \mathcal{S}$.

This can be easily illustrated as discussed below;

$(\mathcal{S}^*\mathcal{S})\ (\mathcal{S}^* + \mathcal{S}) = (\mathcal{S}^* + \mathcal{S})(\mathcal{S}^*\mathcal{S})$. So, $(\mathcal{S}^*\mathcal{S})(\mathcal{S}^* + \mathcal{S}) = (\mathcal{S}^*\mathcal{S}\mathcal{S}^* + \mathcal{S}^*\mathcal{S}\mathcal{S})$.

Let $\mathcal{S}^*\mathcal{S} = A$. Then we have

$$\begin{aligned} \mathcal{S}^*\mathcal{S}\mathcal{S}^* + \mathcal{S}^*\mathcal{S}\mathcal{S} &= A\mathcal{S}^* + A\mathcal{S} \\ &= (\mathcal{S}^* + \mathcal{S})A \\ &= (\mathcal{S}^* + \mathcal{S})(\mathcal{S}^*\mathcal{S}). \end{aligned}$$

An example to this is as follows;

Example 4.2.1. Let $\mathcal{S} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ be an operator on the two-dimensional Hilbert space H^2.

Show that $\mathcal{S}$ is a θ-operator.

Solution: Since $\mathcal{S}$ is self-adjoint, then $\mathcal{S}^* = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. For θ-operator property, we need to establish if $(\mathcal{S}^*\mathcal{S})\ (\mathcal{S}^* + \mathcal{S}) = (\mathcal{S}^* + \mathcal{S})(\mathcal{S}^*\mathcal{S})$. So,

$$(\mathcal{S}^*\mathcal{S}) = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix} \qquad \text{And}$$

$$(\mathcal{S}^* + \mathcal{S}) = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix}.$$

Thus, $(\mathcal{S}^*\mathcal{S})(\mathcal{S}^* + \mathcal{S}) = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix}$

$$= \begin{pmatrix} 42 & 68 \\ 68 & 110 \end{pmatrix} \quad \text{and}$$

$$(\mathcal{S}^* + \mathcal{S})(\mathcal{S}^*\mathcal{S}) = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$$

$$= \begin{pmatrix} 42 & 68 \\ 68 & 110 \end{pmatrix}.$$

Hence, $(\mathcal{S}^*\mathcal{S})(\mathcal{S}^* + \mathcal{S}) = (\mathcal{S}^* + \mathcal{S})(\mathcal{S}^*\mathcal{S})$.

Remark 4.2.2. θ-operators form a ring since they are defined by two operations.

Remark 4.2.3. The following lemma shows that normal operators are θ-operators.

Lemma 4.2.4. If an operator $\mathcal{T} \in \mathfrak{B}(H)$ is normal, it is also a θ-operator.

Proof: Assuming $\mathcal{T}$ is normal, then $\mathcal{T} = \mathcal{T}\mathcal{T}^*\mathcal{T}$. From the property of θ-operator, now we have $\mathcal{T}^*\mathcal{T} = \mathcal{T}^*\mathcal{T}(\mathcal{T}^* + \mathcal{T})$

$$= \mathcal{T}^*\mathcal{T}\mathcal{T}^* + \mathcal{T}^*\mathcal{T}\mathcal{T} \tag{5}$$

Again,

$$(\mathcal{T}^* + \mathcal{T})\mathcal{T}^*\mathcal{T} = \mathcal{T}^*\mathcal{T}^*\mathcal{T} + \mathcal{T}\mathcal{T}^*\mathcal{T} \tag{6}$$

From R.H.S of equation (5), we have

$\mathcal{T}^*\mathcal{T}\mathcal{T}^* + \mathcal{T}^*\mathcal{T}\mathcal{T} = \mathcal{T}^*\mathcal{T}^*\mathcal{T} + \mathcal{T}^*\mathcal{T}\mathcal{T}$

$= {\mathcal{T}^*}^2\mathcal{T} + \mathcal{T}^*\mathcal{T}^2$ (since $\mathcal{T}$ and $\mathcal{T}^*$ commute)

$= {\mathcal{T}^*}^2\mathcal{T} + \mathcal{T}\mathcal{T}^*\mathcal{T}$,

which is similar to the R.H.S of equation (6).

Therefore, every normal operator is a θ-operator. ■

Theorem 4.2.5. (Arun, 1975). If an operator $\mathcal{S} \in \mathfrak{B}(H)$ is an idempotent, then $\mathcal{S}$ is self-adjoint.

Proof: It's first shown that the null space of $\mathcal{S}$, denoted by, $N(\mathcal{S})$ reduces $\mathcal{S}$ and is invariant under $\mathcal{S}^*$. For $\mathcal{S} \in \mathfrak{B}(H)$ then $\mathcal{S}^2 = \mathcal{S}$. It follows that, $\mathcal{S}^*\mathcal{S}\mathcal{S}^* = \mathcal{S}\mathcal{S}^*\mathcal{S}$, thus for $y \in N(\mathcal{S})$, we have $\mathcal{S}^*\mathcal{S}\mathcal{S}^*y = 0$. Then $(\mathcal{S}\mathcal{S}^*)^{\frac{1}{2}}y = 0$. Since $\mathcal{S}\mathcal{S}^* > 0$, $\mathcal{S}\mathcal{S}^*y = 0 \Rightarrow \mathcal{S}^*y \in N(\mathcal{S})$.

Thus, under $\mathcal{S}^*$, $N(\mathcal{S})$ is invariant.

It follows the assertion that $\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$. Therefore, we have $\mathcal{S}(I - \mathcal{S})y = 0$ for $y \in H$, since $\mathcal{S}^2 = \mathcal{S}$. Thus, $(I - \mathcal{S})y \in N(\mathcal{S})$. Hence, $\mathcal{S}^*(I - \mathcal{S})y \in N(\mathcal{S})$ as $N(\mathcal{S})$ reduces $\mathcal{S}$. Therefore, $\mathcal{S}\mathcal{S}^*(\mathrm{I} - \mathcal{S})y = 0$. Thus, $\mathcal{S}^*(\mathrm{I} - \mathcal{S})y = 0$ as $N(\mathcal{S}\mathcal{S}^*) = N(\mathcal{S}^*)$.

Hence, $\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$. ■

Theorem 4.2.6. (Campbell, 1972). Let $A, B \in \mathfrak{B}(H)$. If A is normal operator and B is a projection, then the following conditions hold;

(i) $A^*(I - B) = (I - B)A^*(I - B)(I - B), \{BA^*(I - B) = 0\}$

(ii) $AB = BAB$, i.e $(I - B)AB = 0$

(iii) $B^*(A - A^*)(I - B) = 0$

Proof: If A and B satisfy the above three conditions, then by third and first condition we have;

$$\begin{aligned} B^*A^{*2}(I - B) &= B^*A^*(I - B)A^*(I - B) \\ &= B^*A(I - B)A^*(I - B) \\ &= B^*AA^*(I - B) \end{aligned}$$

Supposing that

$$\mathcal{S} = AB + A^*(I - B) \tag{7}$$

Then $\mathcal{S} \in \theta$. Thus, it follows that;

$$\begin{aligned} \mathcal{S} + \mathcal{S}^* &= AB + A^*(I - B) + B^*A^* + (I - B)A \\ &= A^* + A + [AB - A^*B + B^*A^* - B^*A] \end{aligned}$$

But,

$$\begin{aligned} AB - A^*B + B^*A^* - B^*A &= (A - A^*)B + B^*(A^* - A) \\ &= (A^* - A)B + B^*(A^* - A)B \\ &= (I - B^*)(A - A^*)B \\ &= 0 \end{aligned}$$

Thus, $\mathcal{S} + \mathcal{S}^* = A + A^* \Rightarrow \mathcal{S}^* = A + A^* - \mathcal{S}$, or

$$\mathcal{S}^* = A^*B + A(I - B) \tag{8}$$

Using (7) and (8), it yields;

$\mathcal{S}^*\mathcal{S} = [A^*B + A(I - B)][BAB + (I - B)A^*(I - B)] = A^*A$. Hence, $\mathcal{S} \in \theta$. ■

Theorem 4.2.7. (Campbell, 1972). Let $A \in \mathfrak{B}(H)$, $\mathcal{S} \in \theta$ and $\sigma(\mathcal{S}) \cap \mathcal{R} = \emptyset$. If B is a projection obtained by integrating $(\lambda - \mathcal{S})^{-1}$ in the upper half-plane, then A and B satisfy the conditions in theorem 4.2.6 below and $\mathcal{S} = AB + A^*(I - B)$.

Proof: Since $(\lambda - \mathcal{S}^*)(\lambda - \mathcal{S}) = (\lambda - A^*)(\lambda - A) \forall \lambda \notin \sigma(\mathrm{A}) \cup \sigma(A^*)$, then we have $(A - A^*)B = (\lambda - \mathcal{S})^{-1} = [((\lambda - A)^{-1}) - (\lambda - A^*)^{-1}](\lambda - \mathcal{S}^*)$. By integrating on the upper portion of $\sigma(\mathcal{S})$ and consequently on the lower portion of $\sigma(\mathcal{S})$, it yields

$A - A^* = (A - \mathcal{S}^*)$ or

$B = (A - A^*)^{-1}(A - \mathcal{S}^*)$ and $(A - A^*)(I - B) = -(A^* - \mathcal{S}^*)$ or

$I - B = (A - A^*)^{-1}(\mathcal{S}^* - A^*)$.

From definition of B, we have $\mathcal{S}B = B\mathcal{S}$. It then follows that,

$$\begin{aligned} AB &= A(A - A^*)^{-1}(\mathcal{S} - A^*) \\ &= (A - A^*)^{-1}(A\mathcal{S} - A^*A) \\ &= (A - A^*)^{-1}(A - \mathcal{S}^*)\mathcal{S} \\ &= B\mathcal{S} \\ &= \mathcal{S}B. \end{aligned}$$

Thus, condition (ii) of theorem 4.2.6 hold.

Similarly, $A^*(I - B) = (I - B)\mathcal{S} = \mathcal{S}(I - B)$.

Thus, $\mathcal{S} = AB + A^*(I - B)$ and for condition (iii) we have;

$$\begin{aligned} B^*(A - A^*)(I - B) &= (A^* - \mathcal{S})(A^* - A)^{-1}(A - A^*)(I - B) \\ &= -A^*(I - B) + \mathcal{S}(I - B) \\ &= 0. \end{aligned}$$

■

Theorem 4.2.8 (Campbell, 1980). Two operators $A, B \in \mathfrak{B}(H)$ satisfying conditions of theorem 4.2.6 and B be a projection onto $\mathcal{R}_1$ along $\mathcal{R}_2$, for $\mathcal{R}_1$ being invariant space of A and

$\mathcal{R}_2 = (A - A^*)^{-1}$, such that $\sigma(\mathrm{A}) \subseteq UHP$, then $\mathcal{S} = AB + A^*(I - B) \in \theta$.

Proof: It is clear that $\mathcal{R}_1$ is A invariant, and thus $\mathcal{R}_2$ is A^* invariant since $\mathcal{R}_1^{\perp}$ is A^* invariant. Suppose $(A - A^*)^{1/2}$ denote the square root $(A - A^*)$, then

$(A - A^*)^{\frac{1}{2}} \mathcal{R}_1 \oplus (A - A^*)^{-\frac{1}{2}} \mathcal{R}_1^{\perp} = H$. On multiplying by $(A - A^*)^{-\frac{1}{2}}$, we obtain;

$\mathcal{R}_1 + \mathcal{R}_2 = H$ (direct sum). This shows that B is a bounded and thus the first and second condition of 4.2.6 hold while third condition is equivalent to $(A - A^*)\, \mathcal{R}_2 \subseteq \mathcal{R}_1^{\perp}$. ■

4.3. Unitary equivalence of subclasses of $\theta -$ operators

In this subsection, some results on subclasses of $\theta -$ operators are outlined in the following theorems:

Theorem 4.3.1. Let $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ be such that $\mathcal{T}$ is a θ-operator and $\mathcal{S} = U\mathcal{T}U^*$ where U is an isometry. Then $\mathcal{S}$ is also a θ-operator.

Proof: Since $\mathcal{T}$ is a θ-operator, then

$\mathcal{T}^*\mathcal{T}(\mathcal{T}^* + \mathcal{T}) = (\mathcal{T}^* + \mathcal{T})\mathcal{T}^*\mathcal{T}$ i.e

$$\mathcal{T}^*\mathcal{T}\mathcal{T}^* + \mathcal{T}^*\mathcal{T}^2 = \mathcal{T}^{*2}\mathcal{T} + \mathcal{T}\mathcal{T}^*\mathcal{T} \tag{9}$$

But $\mathcal{S} = U\mathcal{T}U^*$ thus $\mathcal{S}^* = U\mathcal{T}^*U^*$

Therefore, $\mathcal{S}^*\mathcal{S} = U\mathcal{T}^*U^*U\mathcal{T}U^* = U\mathcal{T}^*\mathcal{T}U^*$ and $\mathcal{S}^* + \mathcal{S} = U\mathcal{T}^*U^* + U\mathcal{T}\mathcal{T}U^*$. Thus we have,

$$\mathcal{S}^*\mathcal{S}(\mathcal{S}^* + \mathcal{S}) = U\mathcal{T}^*\mathcal{T}U^*(U\mathcal{T}^*U^* + U\mathcal{T}U^*)$$

$$= U\mathcal{T}^*\mathcal{T}\mathcal{T}^*U^* + U\mathcal{T}^*\mathcal{T}^2U^* \tag{10}$$

Also, $(\mathcal{S}^* + \mathcal{S})\mathcal{S}^*\mathcal{S} = (U\mathcal{T}^*U^* + U\mathcal{T}U^*)U\mathcal{T}^*\mathcal{T}U^*$

$$= U\mathcal{T}^{*2}\mathcal{T}U^* + U\mathcal{T}\mathcal{T}^*\mathcal{T}U^* \tag{11}$$

Using (9) we have that, $U\mathcal{T}^*\mathcal{T}\mathcal{T}^*U^* + U\mathcal{T}^*\mathcal{T}^2U^* = U\mathcal{T}^{*2}\mathcal{T}U^* + U\mathcal{T}\mathcal{T}^*\mathcal{T}U^*$.

From (10) and (11) we have $\mathcal{S}^*\mathcal{S}(\mathcal{S}^* + \mathcal{S}) = (\mathcal{S}^* + \mathcal{S})\mathcal{S}^*\mathcal{S}$. Hence, $\mathcal{S}$ is also a

θ-operator. ■

Theorem 4.3.2. Let $\mathcal{S},\mathcal{T} \in \mathfrak{B}(H)$ be such that $\mathcal{T}$ is a θ-operator and $\mathcal{S} = U^*\mathcal{T}U$ where U is a co-isometry. Then $\mathcal{S}$ is also a θ-operator.

Proof: Since $\mathcal{T}$ is a θ-operator we have that;

$$\mathcal{T}^*\mathcal{T}(\mathcal{T}^* + \mathcal{T}) = (\mathcal{T}^* + \mathcal{T})\mathcal{T}^*\mathcal{T}$$

$$\mathcal{T}^*\mathcal{T}\mathcal{T}^* + \mathcal{T}^*\mathcal{T}^2 = \mathcal{T}^{*2}\mathcal{T} + \mathcal{T}\mathcal{T}^*\mathcal{T} \tag{12}$$

But $\mathcal{S} = U^*\mathcal{T}U$ implies $\mathcal{S}^* = U^*\mathcal{T}^*U$. Therefore,

$$\mathcal{S}^*\mathcal{S} = U^*\mathcal{T}^*UU^*\mathcal{T}U$$

$$= U^*\mathcal{T}^*\mathcal{T}U \quad \text{and}$$

$$\mathcal{S}^* + \mathcal{S} = U^*\mathcal{T}^*U + U^*\mathcal{T}U.$$

Thus we have, $\mathcal{S}^*\mathcal{S}(\mathcal{S}^* + \mathcal{S}) = U^*\mathcal{T}^*\mathcal{T}U(U^*\mathcal{T}^*U + U^*\mathcal{T}U)$

$$= U^*\mathcal{T}^*\mathcal{T}\mathcal{T}^*U + U^*\mathcal{T}^*\mathcal{T}^2U \tag{13}$$

and

$$(\mathcal{S}^* + \mathcal{S})\mathcal{S}^*\mathcal{S} = (U^*\mathcal{T}^*U + U^*\mathcal{T}U)U^*\mathcal{T}^*\mathcal{T}U$$

$$= U^*\mathcal{T}^{*2}\mathcal{T}U + U^*\,\mathcal{T}\,\mathcal{T}^*\mathcal{T}U \tag{14}$$

From (12) we have that;

$U^*\,\mathcal{T}^*\mathcal{T}\,\mathcal{T}^*U + U^*\,\mathcal{T}^*\mathcal{T}^{*2}U = U^*\mathcal{T}^{*2}\mathcal{T}U + U^*\mathcal{T}\mathcal{T}^*\mathcal{T}U$. Hence, from (13) and (14) we have;

$\mathcal{S}^*\mathcal{S}(\mathcal{S}^* + \mathcal{S}) = (\mathcal{S}^* + \mathcal{S})\mathcal{S}^*\mathcal{S}$. Thus $\mathcal{S} \in \theta$. ■

Corollary 4.3.3. Let $\mathcal{S},\mathcal{T} \in \mathfrak{B}(H)$ be such that $\mathcal{T}$ is a θ-operator and either $\mathcal{S} = U^*\mathcal{T}U$ or $\mathcal{S} = U\mathcal{T}U^*$, where U is unitary .Then $\mathcal{S}$ is also a θ-operator.Thus $\mathcal{S}$ and $\mathcal{T}$ are unitarily equivalent θ-operators and hence have equal spectra.

Proof: From the inclusions of classes of operators we note that every unitary operator is either an isometry or co-isometry. Hence by both theorems 4.3.1 and 4.3.2 above the result follows easily. ■

Theorem 4.3.4. (Campbell, 1980). Let $\mathcal{S},\mathcal{T} \in \mathfrak{B}(H)$. If $\mathcal{S}$ is normal and it is unitarily equivalent to $\mathcal{T}$, then $\mathcal{T}$ is normal.

Proof: Since $\mathcal{S}$ is normal, then $\mathcal{T} = U^*\mathcal{S}U$, where U is unitary operator. It then follows that;

$$\mathcal{T}^*\mathcal{T} = (U^*\mathcal{S}^*U)(U^*\mathcal{S}U) = U^*\mathcal{S}^*\mathcal{S}U = U^*\mathcal{S}\mathcal{S}^*U = \mathcal{T}U^*\mathcal{S}^*U = \mathcal{T}U^*U\mathcal{T}^* = \mathcal{T}\mathcal{T}^*.$$

This proves the claim. ■

Theorem 4.3.5. Let $\mathcal{S},\mathcal{T} \in \mathfrak{B}(H)$. If $\mathcal{S}$ is unitarily equivalent to $\mathcal{T}$, denoted by $\mathcal{S} \cong \mathcal{T}$ and $\mathcal{S}$ is a θ-operator, then so is $\mathcal{T}$.

Proof: Since $\mathcal{S}$ is unitarily equivalent to $\mathcal{T}$, there exist a unitary operator U, such that

$\mathcal{S}U = \mathcal{T}U$. i.e, $\mathcal{S} = U^*\mathcal{T}U$ and $\mathcal{S}^* = U^*\mathcal{T}^*U$. Thus,

$$\begin{aligned}(\mathcal{S}^*\mathcal{S}) &= (U^*\mathcal{T}^*U)(U^*\mathcal{T}U)\\ &= U^*\mathcal{T}^*UU^*\mathcal{T}U\\ &= U^*\mathcal{T}^*\mathcal{T}U\\ &= U^*\mathcal{T}^*U\mathcal{T}\\ &= U^*U(\mathcal{T}^*\mathcal{T}), \text{ But } U^*U = I, \text{ hence,}\end{aligned}$$

$$(\mathcal{S}^*\mathcal{S}) = \mathcal{T}^*\mathcal{T} \tag{15}$$

and

$$\begin{aligned}(\mathcal{S}^* + \mathcal{S}) &= U^*\mathcal{T}^*U + U^*\mathcal{T}U\\ &= U^*U\mathcal{T}^* + U^*U\mathcal{T}\\ &= U^*U(\mathcal{T}^* + \mathcal{T}). \text{ Hence,}\end{aligned}$$

$$\mathcal{S}^* + \mathcal{S} = \mathcal{T}^* + \mathcal{T} \tag{16}$$

Again, $\mathcal{S}^*\mathcal{S}(\mathcal{S}^* + \mathcal{S}) = \mathcal{S}^*\mathcal{S}\mathcal{S}^* + \mathcal{S}^*\mathcal{S}\mathcal{S}$

$$= U^*\mathcal{T}^*UU^*\mathcal{T}UU^*\mathcal{T}^*U + U^*\mathcal{T}^*UU^*\mathcal{T}UU^*\mathcal{T}U$$

$$= U^*\mathcal{T}^*\mathcal{T}\mathcal{T}^*U + U^*\mathcal{T}^*\mathcal{T}\mathcal{T}U$$

$$= U^*U\mathcal{T}^*\mathcal{T}\mathcal{T}^* + U^*U\mathcal{T}^*\mathcal{T}\mathcal{T}$$

$$= U^*U(\mathcal{T}^* + \mathcal{T}) \tag{17}$$

and

$$(\mathcal{S}^* + \mathcal{S})\mathcal{S}^*\mathcal{S} = \mathcal{S}^*\mathcal{S}^*\mathcal{S} + \mathcal{S}\mathcal{S}^*\mathcal{S}$$

$$= U^*\mathcal{T}^*UU^*\mathcal{T}^*UU^*\mathcal{T}U + U^*\mathcal{T}UU^*\mathcal{T}^*UU^*\mathcal{T}U$$

$$= U^*\mathcal{T}^*\mathcal{T}^*\mathcal{T}U + U^*\mathcal{T}\mathcal{T}^*\mathcal{T}U$$

$$= U^*\mathcal{T}^*U + U^*\mathcal{T}U$$

$$= U^*U(\mathcal{T}^* + \mathcal{T}) \tag{18}$$

From (15) and (16) and comparing the R.H.S of equation (17) and the R.H.S of equation (18), they are equal. Hence, $\mathcal{T}$ is also a θ-operator. ■

4.4. Almost similarity of $\theta -$ operators

Proposition 4.4.1. (Musundi *et al*., 2013). For an operator $\mathcal{S} \in \mathfrak{B}(H)$, we have $\mathcal{S} \in \theta$ if and only if $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ for some normal operator $\mathcal{T}$.

Proof: Let $\mathcal{S} \in \theta$. Then we have $4\mathcal{S}^*\mathcal{S} - (\mathcal{S}^* + \mathcal{S})^2 \geq 0$. Therefore,

$\mathcal{T} = \mathcal{S}^* + \mathcal{S} + i\sqrt{4\mathcal{S}^*\mathcal{S} - (\mathcal{S}^* + \mathcal{S})^2/2}$ is normal with $\mathcal{T}^*\mathcal{T} = \mathcal{S}^*\mathcal{S}$ and $\mathcal{T}^* + \mathcal{T} = \mathcal{S}^* + \mathcal{S}$. Then it follows that,

$$\mathcal{S}^*\mathcal{S} = I^{-1}\,\mathcal{T}^*\mathcal{T}I \tag{19}$$

and

$$\mathcal{S}^* + \mathcal{S} = I^{-1}(\mathcal{T}^* + \mathcal{T})I \tag{20}$$

From (19) and (20) it justifies that $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$.

Conversely, let $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$. Then an invertible operator $\mathcal{N}$ exists such that

$\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}\,\mathcal{T}^*\mathcal{T}\mathcal{N}$ and

$\mathcal{S}^* + \mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\mathcal{N}$. It follows that

$$\mathcal{S}^*\mathcal{S}(\mathcal{S}^* + \mathcal{S}) = \mathcal{N}^{-1}\,\mathcal{T}^*\mathcal{T}(\mathcal{T}^* + \mathcal{T})\mathcal{N} \tag{21}$$

and

$$(\mathcal{S}^* + \mathcal{S})\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\,\mathcal{T}^*\mathcal{T}\mathcal{N} \tag{22}$$

Therefore, $\mathcal{T} \in \theta$ since it is normal. Thus, the right hand sides of (21) and (22) are equal.

Hence, $(\mathcal{S}^* + \mathcal{S})\mathcal{S}^*\mathcal{S} = \mathcal{S}^*\mathcal{S}(\mathcal{S}^* + \mathcal{S})$, showing that $\mathcal{S} \in \theta$. ■

Proposition 4.4.2. (Nzimbi *et al*., 2008). If two operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ are such that, $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$, then $(\mathcal{S} + \lambda I) \overset{a.s}{\sim} (\mathcal{T} + \lambda I)\ \forall\, \lambda \in \mathbb{R}$.

Proof: Since $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$, then there exists an invertible operator $\mathcal{N}$ such that

$$\mathcal{S}^* + \mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\mathcal{N} \tag{23}$$

and

$$\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}\mathcal{T}^*\mathcal{T}\mathcal{N} \tag{24}$$

It follows from (23) that,

$\mathcal{S}^* + \mathcal{S} = \mathcal{N}^{-1}\mathcal{T}^*\mathcal{N} + \mathcal{N}^{-1}\mathcal{T}\mathcal{N}, \Rightarrow \mathcal{S}^* + \mathcal{S} + 2\lambda = \mathcal{N}^{-1}\mathcal{T}^*\mathcal{N} + \mathcal{N}^{-1}\mathcal{T}\mathcal{N} + 2\lambda.$

Thus, $(\mathcal{S}^* + \lambda I) + (\mathcal{S} + \lambda I) = \mathcal{N}^{-1}(\mathcal{T}^* + \lambda I)(\mathcal{T}^* + \lambda I)\mathcal{N} + \mathcal{N}^{-1}(\mathcal{T} + \lambda I)\mathcal{N}$

$$= \mathcal{N}^{-1}[(\mathcal{T} + \lambda I)^* + (\mathcal{T} + \lambda I)]\mathcal{N},$$

$$= \mathcal{N}^{-1}[(\mathcal{T}^* + \lambda I) + (\mathcal{T} + \lambda I)]\mathcal{N} \tag{25}$$

Which can be further be simplified to yield;

$$\lambda\mathcal{S}^* + \mathcal{S}\lambda + \lambda^2 = \mathcal{N}^{-1}\lambda\mathcal{T}^*\mathcal{N} + \mathcal{N}^{-1}\lambda\mathcal{T}\mathcal{N} + \mathcal{N}^{-1}\lambda^2\mathcal{T}\mathcal{N} \tag{26}$$

By summing (24) and (26), we obtain

$\mathcal{S}^*\mathcal{S} + \lambda\mathcal{S}^* + \mathcal{S}\lambda + \lambda^2 = \mathcal{N}^{-1}\lambda\mathcal{T}^*\mathcal{N} + \mathcal{N}^{-1}\lambda\mathcal{T}\mathcal{N} + \mathcal{N}^{-1}\lambda^2\mathcal{T}^*\mathcal{N} + \mathcal{N}^{-1}\mathcal{T}^*\mathcal{T}\mathcal{N}.$

This implies that,

$(\mathcal{S}^* + \lambda I) + (\mathcal{S} + \lambda I) = \mathcal{N}^{-1}[(\mathcal{T} + \lambda I)^*(\mathcal{T} + \lambda I)]\mathcal{N}$. It follows that,

$$(\mathcal{S} + \lambda I)^*(\mathcal{S} + \lambda I) = \mathcal{N}^{-1}[(\mathcal{T} + \lambda I)^*(\mathcal{T} + \lambda I)]\mathcal{N} \tag{27}$$

Hence, from (25) and (27), we thus conclude that $(\mathcal{S} + \lambda I) \overset{a.s}{\sim} (\mathcal{T} + \lambda I)$ as desired. ■

Corollary 4.4.3. (Sitati, 2011). If two projection operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ are such that $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ and $(\mathcal{S} + \lambda I) \overset{a.s}{\sim} (\mathcal{T} + \lambda I), \forall\, \lambda \in \mathbb{R}$, then, $\sigma_p(\mathcal{S}) = \sigma_p(\mathcal{T})$.

Proof: $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ implies there exists an invertible operator $\mathcal{N}$ such that

$$\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}\mathcal{T}^*\mathcal{T}\mathcal{N} \tag{28}$$

and

$$\mathcal{S}^* + \mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\mathcal{N} \tag{29}$$

It follows that equation (29) can be expressed as $2\mathcal{S} = \mathcal{N}^{-1}2\mathcal{T}\mathcal{N}$, since $\mathcal{S}^* = \mathcal{S}$ and $\mathcal{T}^* = \mathcal{T}$. Thus, $\mathcal{S} = \mathcal{N}^{-1}\mathcal{T}\mathcal{N}$. i.e, $\mathcal{N}\mathcal{S} = \mathcal{T}\mathcal{N}$, thus, $\sigma_p(\mathcal{S}) = \sigma_p(\mathcal{T})$.

Again, (28) can also be expressed as $\mathcal{S}^2 = \mathcal{N}^{-1}\mathcal{T}^2\mathcal{N}$ since $\mathcal{S}^* = \mathcal{S} = \mathcal{S}^2$ and $\mathcal{T}^* = \mathcal{T} = \mathcal{T}^2, \Rightarrow \mathcal{S} = \mathcal{N}^{-1}\mathcal{T}\mathcal{N}$. It therefore follows that, $\sigma_p(\mathcal{S}) = \sigma_p(\mathcal{T})$. ■

Remark 4.4.4. Corollary 4.4.3 above provides us with a situation where two almost similar operators have equal spectrum.

Proposition 4.4.5. If two unitary operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ are such that $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ and $\mathcal{S}$ is a θ-operator, then $\mathcal{T}$ is also a θ-operator.

Proof: $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ implies that, an invertible operator $\mathcal{N}$ exists such that

$$\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}\mathcal{T}^*\mathcal{T}\mathcal{N} \tag{30}$$

and

$$\mathcal{S}^* + \mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\mathcal{N} \tag{31}$$

From (30), we have

$$\mathcal{S} = \mathcal{S}\mathcal{N}^{-1}\mathcal{T}^*\mathcal{T}\mathcal{N}$$

$$= \mathcal{S}\mathcal{N}^{-1}\mathcal{N}\mathcal{T}^*\mathcal{T}$$

$= \mathcal{S}\mathcal{T}^*\mathcal{T}$ and thus, $\mathcal{S}^* = (\mathcal{S}\mathcal{T}^*\mathcal{T})^* = \mathcal{T}^*\mathcal{T}\mathcal{S}^*$.

Applying the property of θ-operator, we have;

$\mathcal{S}^*\mathcal{S} = \mathcal{T}^*\mathcal{T}\mathcal{S}^*\mathcal{S}\mathcal{T}^*\mathcal{T} = \mathcal{T}^*\mathcal{T}\mathcal{T}^*\mathcal{T} = (\mathcal{T}^*\mathcal{T})^2 = \mathcal{T}^*\mathcal{T}$ (projection property). Also, $\mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\mathcal{N} = \mathcal{S}^* + \mathcal{S} = \mathcal{T}^*\mathcal{T}\mathcal{S}^* + \mathcal{S}\mathcal{T}^*\mathcal{T}$. But $\mathcal{S} \cong \mathcal{T}$, then it implies there exists a unitary operator U, such that $\mathcal{S} = U^*\mathcal{T}U$ and $\mathcal{S}^* = U^*\mathcal{T}^*U$.

Thus, $\mathcal{S}^* + \mathcal{S} = \mathcal{T}^*\mathcal{T}\mathcal{S}^* + \mathcal{S}\mathcal{T}^*\mathcal{T}$

$$= \mathcal{T}^*\mathcal{T}U^*\mathcal{T}^*U + U^*\mathcal{T}U\mathcal{T}^*\mathcal{T}$$

$$= \mathcal{T}^*\mathcal{T}\mathcal{T}^*U^*U + U^*U\mathcal{T}\mathcal{T}^*\mathcal{T}$$

$$= \mathcal{T}^*\mathcal{T}\mathcal{T}^* + \mathcal{T}\mathcal{T}^*\mathcal{T}$$

$= \mathcal{T}^*\mathcal{T}(\mathcal{T}^* + \mathcal{T})$, but $\mathcal{T}^*\mathcal{T} = I$, thus

$$= \mathcal{T}^* + \mathcal{T}.$$

This shows that $\mathcal{T}$ is also a θ-operator. ■

Remark 4.4.6. If $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ are such that $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ and if $\mathcal{S}$ is normal then $\mathcal{T}$ is also normal since normal operators are contained in θ-operators.

4.5 On α −Almost Similarity of θ-Operators

We extend the well-known property of *almost similarity* to another property called $\alpha - almost\ similarity$ in this subsection.

According to Amjad & Laith, (2019), Two operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ are $\alpha - almost\ similar$, denoted as $\mathcal{S} \overset{\alpha}{\approx} \mathcal{T}$ if there exists an invertible operator $\mathcal{N}$, such that conditions

$\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}\mathcal{T}^*\mathcal{T}\mathcal{N}$ and

$\mathcal{S}^* + \alpha\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \alpha\mathcal{T})\mathcal{N}, \forall\, \alpha \in \mathbb{R}$ hold.

Remark 4.5.1. If $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ are $\alpha - almost\ similar$ and if $\alpha = 1$, then $\mathcal{S}$ and $\mathcal{T}$ are almost similar.

Remark 4.5.2. For $\alpha \neq 1$, $almost\ similar$ and $\alpha - almost\ similar$ are independent.

Proposition 4.5.3. (Amjad & Laith, 2019). If two self-adjoint operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ are $similar$, i.e $\mathcal{S} \sim \mathcal{T}$, then $\mathcal{S} \overset{\alpha}{\approx} \mathcal{T}$ for every $\alpha \in \mathbb{R}$.

Proof: $\mathcal{S} \sim \mathcal{T}$ implies there exist an invertible operator $\mathcal{N}$ such that $\mathcal{N}\mathcal{S} = \mathcal{T}\mathcal{N}$

$i.e,\ \mathcal{S} = \mathcal{N}^{-1}\mathcal{T}\mathcal{N}$.

From the property of almost similarity and since $\mathcal{S}$ and $\mathcal{T}$ are self-adjoint, we have

$$\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}\mathcal{T}^*\mathcal{T}\mathcal{N} \tag{32}$$

and

$$\begin{aligned} \mathcal{S}^* + \alpha\mathcal{S} &= \mathcal{S} + \alpha\mathcal{S} \\ &= \mathcal{N}^{-1}\mathcal{T}\mathcal{N} + \alpha\mathcal{N}^{-1}\mathcal{T}\mathcal{N} \\ &= \mathcal{N}^{-1}(\mathcal{T} + \alpha\mathcal{T})\mathcal{N} \\ &= \mathcal{N}^{-1}(\mathcal{T}^* + \alpha\mathcal{T})\mathcal{N} \qquad (33) \end{aligned}$$

Thus, it follows from (32) and (33) that $\mathcal{S} \overset{\alpha}{\approx} \mathcal{T}$. ■

Proposition 4.5.4. (Amjad & Laith, 2019). Two self-adjoint operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ are $\alpha - almost\ similar$ if and only if they are $almost\ similar$ and $\alpha \neq -1$.

Proof: If $\mathcal{S}$ and $\mathcal{T}$ are $\alpha - almost\ similar$, then there exist an invertible operator $\mathcal{N}$ such that,

$$\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}\mathcal{T}^*\mathcal{T}\mathcal{N} \tag{34}$$

and

$$\mathcal{S}^* + \alpha\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \alpha\mathcal{T})\mathcal{N} \tag{35}$$

For $\mathcal{S}$ and $\mathcal{T}$ being self-adjoint, then $\mathcal{S} = \mathcal{S}^*$ and $\mathcal{T} = \mathcal{T}^*$.Therefore, (33) above can be expressed as $(1+\alpha)\mathcal{S} = (1+\alpha)\mathcal{N}^{-1}\mathcal{T}\mathcal{N}$.

By pre-multiplying both sides by $\frac{2}{1+\alpha}$, where $\alpha \neq -1$, it yields $2\mathcal{S} = 2\mathcal{N}^{-1}\mathcal{T}\mathcal{N}$ and thus, $\mathcal{S}$ and $\mathcal{T}$ are similar.

Conversely, assume $\mathcal{S}$ and $\mathcal{T}$ are almost similar, then

$$\mathcal{S} + \mathcal{S}^* = \mathcal{N}^{-1}(\mathcal{T} + \mathcal{T}^*)\mathcal{N} \tag{36}$$

It follows from (34) and (36) that $\mathcal{S}$ and $\mathcal{T}$ are almost similar.

Since $\mathcal{S}$ and $\mathcal{T}$ are self-adjoint then (36) can be expressed as $2\mathcal{S} = 2\mathcal{N}^{-1}\mathcal{T}\mathcal{N}$.

Again pre-multiplying it with $\frac{1+\alpha}{2}$ on both sides, it yields $(1+\alpha)\mathcal{S} = (1+\alpha)\mathcal{N}^{-1}\mathcal{T}\mathcal{N}$, and thus

$\mathcal{S} + \alpha\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T} + \alpha\mathcal{T})\mathcal{N}$ and consequently,

$\mathcal{S}^* + \alpha\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \alpha\mathcal{T})\mathcal{N}$, showing that $\mathcal{S}$ and $\mathcal{T}$ are $\alpha - almost\ similar$. ■

Proposition 4.5.5. (Amjad & Laith, 2019). If $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ are projections such that $\mathcal{S} \overset{\alpha}{\approx} \mathcal{T}$ and $(\mathcal{S} + \lambda I) \overset{\alpha}{\approx} (\mathcal{T} + \lambda I)$, then

$\sigma(\mathcal{S}) = \sigma(\mathcal{T}), \sigma_p(\mathcal{S}) = \sigma_p(\mathcal{T})$ and $\sigma_{ap}(\mathcal{S}) = \sigma_{ap}(\mathcal{T})$.

Proof: For $\mathcal{S} \overset{\alpha}{\approx} \mathcal{T}$, implies there exist an invertible operator $\mathcal{N}$ such that

$$\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})\mathcal{N} \tag{37}$$

and

$$\mathcal{S}^* + \alpha\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \alpha\mathcal{T})\mathcal{N} \tag{38}$$

$\mathcal{S}$ and $\mathcal{T}$ are self-adjoint since they are both projections. Thus (38) can be expressed as $(1+\alpha)\mathcal{S} = \mathcal{N}^{-1}(1+\alpha)\mathcal{T}\mathcal{N}$. Then $\mathcal{S} = \mathcal{N}^{-1}\mathcal{T}\mathcal{N}$, i.e, $\mathcal{S} \sim \mathcal{T}$.

Hence, from [Kipkemboi, (2016), Proposition 2.3.27], it follows that

$\sigma(\mathcal{S}) = \sigma(\mathcal{T}), \sigma_p(\mathcal{S}) = \sigma_p(\mathcal{T})$ and $\sigma_{ap}(\mathcal{S}) = \sigma_{ap}(\mathcal{T})$.

Remark 4.5.6. The result above gives us another situation where two almost similar operators have not only equal spectrum but also equal point and approximate spectra.

Remark 4.5.7. We note that apart from normal operators, there are several other classes which satisfy Putnam-Fugede theorem property which states that, for operators $A, B, \mathcal{V} \in \mathfrak{B}(H)$, where A and B are normal and $A\mathcal{V} = \mathcal{V}B$, then $A^*\mathcal{V} = \mathcal{V}B^*$. For example, M-hyponormal operators, P-hyponormal operators etc.

Theorem 4.5.8. (Amjad & Laith, 2019). Let $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ and $\alpha \in \mathbb{R}$. If $\mathcal{N}\mathcal{S} = \mathcal{T}\mathcal{N}$ and $\mathcal{N}\mathcal{S}^* = \mathcal{T}^*\mathcal{N}$, for an invertible operator $\mathcal{N}$, then $\mathcal{S} \overset{\alpha}{\approx} \mathcal{T}$.

Proof: Supposing that $\mathcal{N}\mathcal{S} = \mathcal{T}\mathcal{N}$ and $\mathcal{N}\mathcal{S}^* = \mathcal{T}^*\mathcal{N}$, then we have $\mathcal{S} = \mathcal{N}^{-1}\mathcal{T}\mathcal{N}$ and $\mathcal{S}^* = \mathcal{N}^{-1}\mathcal{T}^*\mathcal{N}$. It therefore follows that,

$\mathcal{S}^*\mathcal{S} = (\mathcal{N}^{-1}\mathcal{T}^*\mathcal{N})(\mathcal{N}^{-1}\mathcal{T}\mathcal{N}) = \mathcal{N}^{-1}\mathcal{T}^*(\mathcal{N}\mathcal{N}^{-1})\mathcal{T}\mathcal{N} = \mathcal{N}^{-1}\mathcal{T}^*\mathcal{T}\mathcal{N}$ and

$\mathcal{S}^* + \alpha\mathcal{S} = \mathcal{N}^{-1}\mathcal{T}^*\mathcal{N} + \mathcal{N}^{-1}(\alpha\mathcal{T})\mathcal{N} = \mathcal{N}^{-1}(\mathcal{T}^* + \alpha\mathcal{T})\mathcal{N}$.

Hence, $\mathcal{S} \overset{\alpha}{\approx} \mathcal{T}$ as desired.

In the next result, we show how similarity holds for α −almost similar. ■

Corollary 4.5.9. Let $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ and $\mathcal{N}$ be an invertible operator such that $\mathcal{N}\mathcal{S} = \mathcal{T}\mathcal{N}$ where $\mathcal{S}$ and $\mathcal{T}$ satisfy the Putnam-Fuglede property. Then $\mathcal{S}$ and $\mathcal{T}$ are α −almost similar.

Proof: If $\mathcal{S}$ and $\mathcal{T}$ satisfy the Puntam-Fuglede property, then $\mathcal{N}\mathcal{S} = \mathcal{T}\mathcal{N}$ and this implies $\mathcal{N}\mathcal{S}^* = \mathcal{T}^*\mathcal{N}$ and the result follows from Theorem 4.5.8 above. ■

CHAPTER FIVE

ALMOST SIMILARITY PROPERTY ON POSINORMAL OPERATORS

5.1 Introduction.

In this chapter we investigate almost similarity property on posinormal operators together with its subclasses which are as follows:

{Projection operators}⊆{Positive operators}⊆{Self-adjoint operators}⊆

{Normal operators}⊆{Hyponormal }⊆{M-hyponormal}⊆{Dominant }⊆{Posinormal }.

We first consider results on general properties of posinormal operators as proved by various authors in the following subsection.

5.2 General properties of posinormal operators.

Posinormal operators have been studied by some authors. On the very essential characteristics, the following theorems are outlined:

Theorem 5.2.1. (Rhaly, 2013). An interrupter P of a posinormal operator $\mathcal{T}$ is unique if $\mathcal{T}$ has a dense range.

Proof: Considering P_1 and P_2 to serve as interrupters for $\mathcal{T}$, then;

$\mathcal{T}^*P_1\mathcal{T} = \mathcal{T}\mathcal{T}^* = \mathcal{T}^*P_2\mathcal{T}$. Thus, $\mathcal{T}^*(P_1 - P_2)\mathcal{T} = 0$. Since $\mathcal{T}$ has dense range, $\mathcal{T}^*$ is one to one and consequently, $(P_1 - P_2)\mathcal{T} = 0$. By the fact that $\mathcal{T}$ has dense range, then it is easily concluded that, $(P_1 - P_2) = 0$. Hence, with $\mathcal{T}$ having dense range and $\mathcal{S}$ serving as an interrupter for $\mathcal{T}$, then $\mathcal{T}$ is posinormal and the interrupter $\mathcal{S}$ is positive and unique.

Theorem 5.2.2. (Rhaly, 2013). For a posinormal operator $\mathcal{T}$ with interrupter P, we have;

(i) $\mathcal{T}$ is hyponormal if and only if the restriction $I - P$ to $Ran\,\mathcal{T}$ is a positive operator.

(ii) If $\|P\| = 1$, then $\mathcal{T}$ is hyponormal.

Proof; (i) it follows that; $\langle[\mathcal{T}^*\mathcal{T}]\, f, f\rangle = \langle(I-P)\mathcal{T}f, \mathcal{T}f\rangle\ , \forall\ f \in H$.

(ii) $\|P\| = 1$, then also, $\|\sqrt{P}\| = 1$. Thus, $\|\mathcal{T}^*f\| \leqq \|\sqrt{P}\| = \|\mathcal{T}f\|$, which is one of the equivalent conditions for the hyponormality of $\mathcal{T}$. ■

Theorem 5.2.3. (Rhaly, 2013). The following statements are equivalent for $\mathcal{T} \in \mathfrak{B}(H)$;

(i) $\mathcal{T}$ is posinormal.

(ii) $Ran\, \mathcal{T} \subseteq Ran\, \mathcal{T}^*$

(iii) $\mathcal{T}\mathcal{T}^* \leqq \lambda^2\mathcal{T}^*\mathcal{T}$, for some $\lambda \geqq 0$.

(iv) There exist an operator $A \in \mathfrak{B}(H)$ such that $\mathcal{T} = \mathcal{T}^*A$.

Also, if (i), (ii), (iii) and (iv) holds, then a unique operator A exist such that;

(a) $\|A\|^2 = \inf\{\frac{\mu}{TT^*} \leqq \mu TT^*\}$

(b) $Ker\, T = Ker\, A$

(c) $Ran\, A \subseteq \overline{(Ran\, T)}$

Theorem 5.2.4. (Campbell, 1972). An operator $\mathcal{T} \in \mathfrak{B}(H)$ is positive normal if and only if there exist a positive operator P, such that $\mathcal{T}\mathcal{T}^* \leq \mathcal{T}^*P\mathcal{T}$.

Proof: For any $y \in H$ then

$$\begin{aligned}\|\mathcal{T}^*y\|^2 &= \langle\mathcal{T}\mathcal{T}^*y, y\rangle\\ &\leq \langle\mathcal{T}^*P\mathcal{T}y, y\rangle\\ &= \langle\sqrt{P}\mathcal{T}y, \sqrt{P}\mathcal{T}y\rangle\\ &= \|\sqrt{P}\mathcal{T}y\|^2\\ &\leq \|\sqrt{P}\|^2\|\mathcal{T}y\|^2\end{aligned}$$

Hence, for any $y \in H$ we have $\|\mathcal{T}^*y\| \leq \|\sqrt{P}\|\,\|\mathcal{T}y\|$.

Suppose we let $\lambda = \left\|\sqrt{P}\right\|$. Then for any $\mathcal{y} \in H$, we have $\left\|\mathcal{T}^*\mathcal{y}\right\| \leq \lambda\left\|\mathcal{T}\mathcal{y}\right\|$ for some $\lambda \geq 0$. By [Douglas, (1966), theorems 1 and 2], there exist an operator $A \in \mathcal{B}(H)$ such that, $\mathcal{T} = \mathcal{T}^*A$.

Therefore, $\mathcal{T}\mathcal{T}^* = (\mathcal{T}^*A)(A^*\mathcal{T}) = \mathcal{T}^*(AA^*)\mathcal{T}$. Thus, $\mathcal{T}$ is a positive-normal operator with an interrupter AA^*. Taking $P = I$ on this theorem, it implies that every hyponormal operator is positive normal. ■

Proposition 5.2.5. (Campbell, 1972). If $\mathcal{T} \in \mathfrak{B}(H)$ is positive normal, then

$Ker\ (\mathcal{T}) = Ker\ (\mathcal{T}^2)$.

Proof: It suffices to show that $Ker(\mathcal{T}^2) \subset Ker(\mathcal{T})$. If $\mathcal{y} \in Ker(\mathcal{T}^2)$, then $\mathcal{T}^2\mathcal{y} = 0$. Thus,

$\mathcal{T}\mathcal{y} \in Ker(\mathcal{T})$. Since $Ker(\mathcal{T}) \subset Ker(\mathcal{T}^*), then,\ \mathcal{T}\mathcal{y} \in Ker(\mathcal{T}^*)$. Hence,

$\mathcal{T}^*\mathcal{T}\mathcal{y} = 0$. Now,
$$\begin{aligned}\|\mathcal{T}\mathcal{y}\|^2 &= \langle \mathcal{T}\mathcal{y}, \mathcal{T}\mathcal{y}\rangle \\ &= \langle \mathcal{T}^*\mathcal{T}\mathcal{y}, \mathcal{y}\rangle \\ &\leq \|\mathcal{T}^*\mathcal{T}\mathcal{y}\|\ \|\mathcal{y}\| \\ &= 0.\end{aligned}$$

Thus, $\mathcal{T}\mathcal{y} = 0$ and so we have, $\mathcal{y} \in Ker(\mathcal{T})$. ■

Theorem 5.2.6. (Halmos, 1982b). Let $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{B}(H)$ be hyponormal operators. If $\mathcal{S}_1$ commutes with the positive part of $\mathcal{S}_2$ and $\mathcal{S}_2$ commutes with the positive part of $\mathcal{S}_1^*$, then $\mathcal{S}_1\mathcal{S}_2$ and $\mathcal{S}_2\mathcal{S}_1$ are hyponormal.

Proof: Hyponormality of $\mathcal{S}_1\mathcal{S}_2$ is proved first. Assuming $\mathcal{S}_1^*(\mathcal{S}_2^*\,\mathcal{S}_2) = (\mathcal{S}_2^*\,\mathcal{S}_2)\mathcal{S}_1^*$ and $\mathcal{S}_2^*(\mathcal{S}_1\,\mathcal{S}_1^*) = (\mathcal{S}_1\,\mathcal{S}_1^*)\mathcal{S}_2^*$, and since for a positive operator P, Q^*PQ is a positive operator for every operator Q, then we can have;

$$(\mathcal{S}_1\mathcal{S}_2)^*(\mathcal{S}_1\mathcal{S}_2) - (\mathcal{S}_1\mathcal{S}_2)(\mathcal{S}_1\mathcal{S}_2)^* = \mathcal{S}_2^*\,\mathcal{S}_1^*\mathcal{S}_1\mathcal{S}_2 - \mathcal{S}_1\mathcal{S}_2\mathcal{S}_2^*\mathcal{S}_1^*$$

$$\geq \mathcal{S}_2^* \mathcal{S}_1 \mathcal{S}_1^* \mathcal{S}_2 - \mathcal{S}_1 \mathcal{S}_2 \mathcal{S}_2^* \mathcal{S}_1^*$$

$$\geq \mathcal{S}_1 \mathcal{S}_1^* \mathcal{S}_2^* \mathcal{S}_2 - \mathcal{S}_1 \mathcal{S}_2^* \mathcal{S}_2 \mathcal{S}_1^*$$

$$\geq \mathcal{S}_1 \mathcal{S}_1^* \mathcal{S}_2^* \mathcal{S}_2 - \mathcal{S}_1 \mathcal{S}_1^* \mathcal{S}_2^* \mathcal{S}_2$$

$$= 0$$

Next, the hyponormality of $\mathcal{S}_2 \mathcal{S}_1$ is established.

For an operator $A \in \mathfrak{B}(H)$ and $\mathcal{y} \in H$ we have;

$$\|A\mathcal{y}\|^2 \langle A\mathcal{y}, A\mathcal{y} \rangle = \langle A^* A\mathcal{y}, \mathcal{y} \rangle$$

$$= \langle (A^* A)^{\frac{1}{2}} \mathcal{y} \rangle$$

$$\| \mathrm{A}\mathcal{y} \|^2 \langle \mathrm{A}, \mathrm{A}\mathcal{y} \rangle = \langle A^* \mathrm{A}\mathcal{y}, \mathcal{y} \rangle = \langle (A^* \mathrm{A})^{\frac{1}{2}} \mathcal{y}, (A^* \mathrm{A})^{\frac{1}{2}} \mathcal{y} \rangle$$

$$= \| (A^* \mathrm{A})^{\frac{1}{2}} \|^2.$$

Therefore,

$$\| (\mathcal{S}_2 \mathcal{S}_1)^* \mathcal{y} \| = \| \mathcal{S}_1^* \mathcal{S}_2^* \mathcal{y} \|$$

$$= \| (\mathcal{S}_1 \mathcal{S}_1^*)^{\frac{1}{2}} \mathcal{S}_2^* \mathcal{y} \|$$

$$= \| \mathcal{S}_2^* (\mathcal{S}_1 \mathcal{S}_1^*)^{\frac{1}{2}} \mathcal{y} \|$$

$$\leq \| \mathcal{S}_2 (\mathcal{S}_1 \mathcal{S}_1^*)^{\frac{1}{2}} \mathcal{y} \|$$

$$= \| (\mathcal{S}_2^* \mathcal{S}_2)^{\frac{1}{2}} (\mathcal{S}_1 \mathcal{S}_1^*)^{\frac{1}{2}} \mathcal{y} \|$$

$$= \| \mathcal{S}_1^* (\mathcal{S}_2^* \mathcal{S}_2)^{\frac{1}{2}} \mathcal{y} \|$$

$$\leq \| \mathcal{S}_1 (\mathcal{S}_1^* \mathcal{S}_1)^{\frac{1}{2}} \mathcal{y} \|$$

$$= \| \mathcal{S}_2 \mathcal{S}_1 \mathcal{y} \| \qquad \blacksquare$$

Corollary 5.2.7. (Halmos, 1982b). Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{B}(H)$ be normal operators. Then each of $\mathcal{T}_1$ and $\mathcal{T}_2$ commutes with the positive part of the other if and only if $\mathcal{T}_1\ \mathcal{T}_2$ and $\mathcal{T}_2\,\mathcal{T}_1$ are normal.

Proof: Since the positive part of $\mathcal{T}_1$ and $\mathcal{T}_2$ are the same as $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ respectively, it follows that $\mathcal{T}_1\mathcal{T}_2$ and $(\mathcal{T}_1\mathcal{T}_2)^* = \mathcal{T}_2^*\mathcal{T}_1^*$ are both hyponormal operators yielding the normality of $\mathcal{T}_1\mathcal{T}_2$ and the normality of $\mathcal{T}_2\mathcal{T}_1$ follows similarly. Conversely,

let $A, B \in \mathfrak{B}(H)$, such that A and AB are normal, then BA is normal. ■

Corollary 5.2.8 (Halmos, 1982b). Let $\mathcal{T}_j = U_j P_j (\, j = 1,2, \ldots)$ be normal operators in their polar decomposition and suppose that $U_1 P_2 = P_2 U_1$, $U_2 P_1 = P_1 U_2$ and $P_1 P_2 = P_2 P_1$, then $\mathcal{T}_1\mathcal{T}_2$ and $\mathcal{T}_2\mathcal{T}_1$ are normal.

Proof: If $\mathcal{T}_1 P_2 = U_1 P_1$, $P_2 = U_1 P_2$, $P_1 = P_2 U_1$ and $P_1 = P_2 \mathcal{T}_1$.Thus $\mathcal{T}_1$ commutes with the positive part of $\mathcal{T}_2$ and vice versa. Hence, $\mathcal{T}_1\ \mathcal{T}_2$ and $\mathcal{T}_2\,\mathcal{T}_1$ are normal. ■

Corollary 5.2.9. (Halmos,1982b)**.** Let $Q, P \in \mathfrak{B}(H)$ be self-adjoint operators, with P positive.Then the following are equivalent;

(i) $PQ = QP$

(ii) QP is normal.

Proof: If QP is normal, then $(PQ)^* = QP$, thus normal. Since $P(QP) = (PQ)P$, it implies that $P^2 Q = QP^2$. Therefore, by Putman-Fuglede theorem and P being positive, then we have

$PQ = QP$. The converse is obvious. ■

Theorem 5.2.10. (Halmos, 1982a). Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{B}(H)$ be operators with one of them normal. If $\mathcal{T}_1\mathcal{T}_2^* = \mathcal{T}_2^*\mathcal{T}_1$, then each of $\mathcal{T}_1$ and $\mathcal{T}_2$ commutes with positive part of the other, but not conversely.

Proof: If $\mathcal{T}_1$ is normal, then by the Fuglede's theorem, we have the relation $\mathcal{T}_1\mathcal{T}_2^* = \mathcal{T}_2^*\mathcal{T}_1$. Thus it follows that $\mathcal{T}_1^*\mathcal{T}_2^* = \mathcal{T}_2^*\mathcal{T}_1^*$ so that, $\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1$. Thus,

$$\begin{aligned}\mathcal{T}_1(\mathcal{T}_2^*\mathcal{T}_2) &= (\mathcal{T}_1\mathcal{T}_2^*)\mathcal{T}_2\\ &= \mathcal{T}_2^*(\mathcal{T}_1\mathcal{T}_2)\\ &= (\mathcal{T}_2^*\mathcal{T}_1)\mathcal{T}_1. \text{ Also,}\\ \mathcal{T}_2(\mathcal{T}_1^*\mathcal{T}_1) &= (\mathcal{T}_2\mathcal{T}_1^*)\mathcal{T}_1\\ &= \mathcal{T}_1^*(\mathcal{T}_2\mathcal{T}_1)\\ &= (\mathcal{T}_1^*\mathcal{T}_1)\mathcal{T}_2\ .\end{aligned}$$

But the converse doesn't hold as can be seen by taking $\mathcal{T}_1$ and $\mathcal{T}_2$ to any two non-commuting unitary operators. ∎

Theorem 5.2.11 (Rhaly, 1994)**.** Every invertible operator is posinormal.

Proof: If an operator $\mathcal{W} \in \mathfrak{B}(H)$ is invertible, then

$\mathcal{W}^* = \mathcal{W}^*(\mathcal{W}^{-1}\mathcal{W}) = (\mathcal{W}^*\mathcal{W}^{-1})\mathcal{W}$. Thus, $\mathcal{W}^* \in [\mathcal{W}]$. ∎

Corollary 5.2.12. (Rhaly, 1994)**.** Every invertible operator is coposinormal.

Theorem 5.2.13. (Rhaly, 2013). If $\mathcal{S} \in \mathfrak{B}(H)$ is coposinormal, satisfying

$\mathcal{S}^*\mathcal{S}^* = \mathcal{S}Q\mathcal{S}^*$, for some positive operator $Q \in \mathfrak{B}(H)$ and P is a positive operator such that $Q \geq P \geq 1$, then $X \equiv \sqrt{P}\mathcal{S}\sqrt{P}$ is hyponormal.

Proof: Let $[\mathcal{X}^*\mathcal{X}] \equiv \mathcal{X}^*\mathcal{X} - \mathcal{X}\mathcal{X}^*$, then it follows that,

$\langle[\mathcal{X}^*\mathcal{X}]f, f\rangle = \langle(P - I)\mathcal{S}\sqrt{P}f\rangle + \langle(Q - P)\mathcal{S}^*\sqrt{P}f\rangle \geq 0,\ \forall\ f \in H^2$, so $\mathcal{X}$ is hyponormal.

We recall that, an operator $\mathcal{T} \in \mathfrak{B}(H)$ is said to be an M-hyponormal operator if there exist a real number M such that

$\left\|(\mathcal{T} - zI)^*\mathcal{y}\right\| \leq M\left\|(\mathcal{y} - zI)^*\mathcal{y}\right\|, \forall\ \mathcal{y} \in \mathfrak{B}(H)$ and $z \in \mathbb{C}$.

Also notable is that, if $\mathcal{T}$ is an M-hyponormal operator, then $M \geq 1$ and M is hyponormal if and only if $M = 1$. ■

Theorem 5.2.14. (Sitati *et al*., 2015). Let $\mathcal{S} \in \mathfrak{B}(H)$ be an M-hyponormal operator and $P \in \mathfrak{B}(H)$ be a hermitian operator. If $\mathcal{S}P$ is a contraction, then $\|P\mathcal{S}\| \leq M$, where $M \in \mathbb{R}$.

Proof: If $\mathcal{S}$ is M-hyponormal and $\|P\mathcal{S}\| \leq 1$, then $\|\mathcal{S}^*P\mathcal{y}\| \leq M\|\mathcal{y}\|$.

Hence, $\|P\mathcal{S}\| = \|(P\mathcal{S})^*\| = \|\mathcal{S}^*P\| \leq M$. This implies that, for a hyponormal $\mathcal{S}$, then $P\mathcal{S}$ is a contraction. ■

Theorem 5.2.15. (Sitati *et al*., 2015). If $\mathcal{S} \in \mathfrak{B}(H)$ is an M-hyponormal then, there exists operators $A, B \in \mathfrak{B}(H)$ which satisfy the following:

(i) $B \geq A \geq 0$,

(ii) $\|\mathcal{S}\| \leq M, M \geq 1$,

(iii) $\mathcal{S}^*A\mathcal{S} = M$. Thus, $\mathcal{S}$ can be expressed as $\mathcal{S} = \frac{1}{M}(A^{\frac{1}{2}} + \lambda I)$, for some complex number λ.

Proof: Let $\mathcal{S}$ be M-hyponormal operator taking $\mathcal{S} - zI = \mathcal{W}$, where z is any complex number. Let $\mathcal{W}^* = U(\mathcal{W}\mathcal{W}^*)^{\frac{1}{2}}$ be polar decomposition of $\mathcal{W}^*$. If $A = \mathcal{W}\mathcal{W}^*$ and $B = M^2\mathcal{W}^*\mathcal{W}$, then we have $B \geq A \geq 0$, since $\mathcal{S}$ is M-hyponormal. Also,

$B = M^2\mathcal{W}^*\mathcal{W} = M^2U(\mathcal{W}\mathcal{W}^*)^{\frac{1}{2}}(\mathcal{W}\mathcal{W}^*)^{\frac{1}{2}}U^* = M^2U\mathcal{W}\mathcal{W}^*U^* = M^2UAU^*$.

Let $\mathcal{T} = MU^*$. Then $\|\mathcal{T}\| \leq M$ and $B = \mathcal{T}^*A\mathcal{T}$. Now,

$\mathcal{S} = V + zI = (\mathcal{W}\mathcal{W}^*)^{\frac{1}{2}}U^* + zI = (\frac{I}{M})A^{\frac{1}{2}}\mathcal{T} + zI = \frac{I}{M}(A^{\frac{1}{2}}\mathcal{T} + zI)$, where λ is some complex number. This establishes the proof. ■

Theorem 5.2.16 (Veluchamy & Thulasimani, 2010). Let $A, B, \mathcal{S} \in \mathfrak{B}(H)$ be operators, then the operator $\mathcal{T} = A^{\frac{1}{2}}\mathcal{S}$ is a positive-normal operator if the following holds;

$(i) B \geq A \geq 0$

$(ii) \|\mathcal{S}\| \leq 1$

(iii) $B = \mathcal{C}^2\mathcal{S}^*A\mathcal{S}$, for $\mathcal{C} > 0$.

Proof: By definition, an operator $\mathcal{T} \in \mathfrak{B}(H)$ is posinormal if $\mathcal{T}\mathcal{T}^* \leq \mathcal{C}^2\mathcal{T}^*\mathcal{T}$, where $\mathcal{C} > 0$. If there exits operators $A, B, \mathcal{S} \in \mathfrak{B}(H)$ which satisfies the above conditions, then;

$$\begin{aligned}
\mathcal{C}^2(A^{\frac{1}{2}}\mathcal{S})^*(A^{\frac{1}{2}}\mathcal{S}) - (A^{\frac{1}{2}}\mathcal{S})(A^{\frac{1}{2}}\mathcal{S})^* &= \mathcal{C}^2(\mathcal{S}^*A^{\frac{1}{2}}A^{\frac{1}{2}}\mathcal{S}) - (A^{\frac{1}{2}}\mathcal{S}\mathcal{S}^*A^{\frac{1}{2}}) \\
&= \mathcal{C}^2\mathcal{S}^*A\mathcal{S} - A^{\frac{1}{2}}\mathcal{S}\mathcal{S}^*A^{\frac{1}{2}} \\
&= B - A^{\frac{1}{2}}\mathcal{S}\mathcal{S}^*A^{\frac{1}{2}} \\
&\geq A - A^{\frac{1}{2}}\mathcal{S}\mathcal{S}^*A^{\frac{1}{2}} \\
&= A^{\frac{1}{2}}(I - \mathcal{S}\mathcal{S}^*)A^{\frac{1}{2}} \\
&\geq 0.
\end{aligned}$$

Thus,

$\mathcal{C}^2(A^{\frac{1}{2}}\mathcal{S})^*(A^{\frac{1}{2}}\mathcal{S}) \geq (A^{\frac{1}{2}}\mathcal{S})(A^{\frac{1}{2}}\mathcal{S})^*$, implying that $A^{\frac{1}{2}}\mathcal{S}$ is positive-normal. ■

Theorem 5.2.17. (Veluchamy & Thulasimani, 2010). An operator $\mathcal{T} \in \mathfrak{B}(H)$ such that $\mathcal{T} = A^{\frac{1}{2}}\mathcal{S}$ and satisfying the following;

(i) $B \geq A \geq 0$

(ii) $\|\mathcal{S}\| \leq 1$

(iii) $B = \mathcal{C}^2\mathcal{S}^*A\mathcal{S}$, for $\mathcal{C} > 0$, is coposinormal, thus $\mathcal{T}^*$ is positive normal.

Proof: If $\mathcal{T}^*\mathcal{T} \leq \mathcal{C}^2\mathcal{T}\mathcal{T}^*$, it follows that $\mathcal{T}$ is posinormal. Thus,

$$\mathcal{C}^2\mathcal{T}\mathcal{T}^* - \mathcal{T}^*\mathcal{T} = \mathcal{C}^2(A^{\frac{1}{2}}\mathcal{S})(A^{\frac{1}{2}}\mathcal{S})^* - (A^{\frac{1}{2}}\mathcal{S})^*(A^{\frac{1}{2}}\mathcal{S})$$

$$= \mathcal{C}^2A^{\frac{1}{2}}\mathcal{S}\mathcal{S}^*A^{\frac{1}{2}} - \mathcal{S}^*A\mathcal{S}$$

$$= \mathcal{C}^2A^{\frac{1}{2}}\mathcal{S}\mathcal{S}^*A^{\frac{1}{2}} - \frac{B}{\mathcal{C}^2}$$

$$= \mathcal{C}^2A^{\frac{1}{2}}\mathcal{S}\mathcal{S}^*A^{\frac{1}{2}} - A$$

$$= A^{\frac{1}{2}}(\mathcal{C}^2\mathcal{S}\mathcal{S}^* - I)A^{\frac{1}{2}}$$

$$\geq 0.$$

This shows that, $\mathcal{T}$ is posinormal. ∎

Theorem 5.2.18. (Veluchamy & Thulasimani, 2010). Let $A, B, \mathcal{S} \in \mathfrak{B}(H)$ be normal operators. If $\mathcal{T} \in \mathfrak{B}(H)$ such that $\mathcal{T} = A^{\frac{1}{2}}\mathcal{S}$ and satisfies the following;

(i) $B \geq A \geq 0$

(ii) $\|\mathcal{S}\| \leq 1$

(iii) $B = \mathcal{C}^2\mathcal{S}^*A\mathcal{S}$, for $\mathcal{C} > 0$, is heminormal.

Proof: From definition, a normal operator, $\mathcal{T} \in \mathfrak{B}(H)$ is heminormal if $\mathcal{T}$ is hyponormal and $\mathcal{T}^*\mathcal{T}$ commutes with $\mathcal{T}\mathcal{T}^*$. Thus,

$$\mathcal{T}^*\mathcal{T}\mathcal{T}\mathcal{T}^* = (A^{\frac{1}{2}}\mathcal{S})^*(A^{\frac{1}{2}}\mathcal{S})(A^{\frac{1}{2}}\mathcal{S})(A^{\frac{1}{2}}\mathcal{S})^*$$

$$= \mathcal{S}^*A\mathcal{S}A^{\frac{1}{2}}\mathcal{S}\mathcal{S}^*A^{\frac{1}{2}}$$

$$= \frac{B}{\mathcal{C}^2}A^{\frac{1}{2}}\mathcal{S}\mathcal{S}^*A^{\frac{1}{2}}$$

and

$$\mathcal{T}\mathcal{T}^*\mathcal{T}^*\mathcal{T} = (A^{\frac{1}{2}}\mathcal{S})(A^{\frac{1}{2}}\mathcal{S})^*(A^{\frac{1}{2}}\mathcal{S})^*(A^{\frac{1}{2}}\mathcal{S})$$

$$= A^{\frac{1}{2}}\mathcal{S}\mathcal{S}^*A^{\frac{1}{2}}\mathcal{S}^*A\mathcal{S}$$

$$= A^{\frac{1}{2}}\mathcal{S}\mathcal{S}^* A^{\frac{1}{2}} \frac{B}{\mathcal{C}^2}$$

This shows that, $\mathcal{T}^*\mathcal{T}\mathcal{T}\mathcal{T}^* = \mathcal{T}\mathcal{T}^*\mathcal{T}^*\mathcal{T}$. Hence, $\mathcal{T}$ is herminormal. ■

Theorem 5.2.19 (Rhaly, 1994). Let an operator $\mathcal{S} \in \mathfrak{B}(H)$ be a posinormal with interrupter P, then for every $\lambda \neq 0$, we have;

(i) $\lambda\mathcal{S}$ is posinormal.

(ii) The translate $\mathcal{S} + \lambda$ need not be posinormal.

Proof: (i) $(\lambda\mathcal{S})(\lambda\mathcal{S})^* = |\lambda|^2\mathcal{S}\mathcal{S}^* = \mathcal{S}^*P\mathcal{S} = (\lambda\mathcal{S})^*P(\lambda\mathcal{X})$, where P is an interrupter.

(ii) On the case where $\mathcal{S} = U^* - 2$ and $\lambda = 2$, (where U^* is the adjoint of unilateral shift), then According to [Halmos,(1982), problem 82], it is established that $2 \notin \sigma(U^*)$. Thus $\mathcal{S}$ is posinormal but $\mathcal{S} + 2 = U^*$ is not posinormal.

We recall that for a posinormal operator $\mathcal{S} \in \mathfrak{B}(H)$ the posispectrum of $\mathcal{S}$ denoted by $\sigma_{po}(\mathcal{S})$ is the set $\sigma_{po}(\mathcal{S}) = \{\lambda : \lambda I - \mathcal{S} \text{ is not posinormal}\}$. ■

5.3. Unitary Equivalence on Subclasses of Posinormal Operators.

Rhaly (1994), in his paper on posinormal operators noted without proof that, for a posinormal operator A with an interrupter P and an isometry $\mathcal{W}$, then it can be checked that $\mathcal{W}^*A\mathcal{W}$ is posinormal with interrupter $\mathcal{W}P\mathcal{W}^*$ and that posinormality is a unitary invariant. We consequently provide proof of this in the following results:

Theorem 5.3.1. Let $\mathcal{T} \in \mathfrak{B}(H)$ be posinormal and $\mathcal{S}$ be any operator such that

$\mathcal{T} = \mathcal{W}\mathcal{S}\mathcal{W}^*$ where $\mathcal{W}$ is an isometry. Then $\mathcal{S}$ is also posinormal.

Proof: Since $\mathcal{T}$ is posinormal, we have $\mathcal{T}\mathcal{T}^* = \mathcal{T}^*P\mathcal{T}$, where P is an interrupter.

From $\mathcal{T} = \mathcal{W}\mathcal{S}\mathcal{W}^*$, it follows that $\mathcal{T}^* = (\mathcal{W}\mathcal{S}\mathcal{W}^*)^*$

$$= \mathcal{W}\mathcal{S}^*\mathcal{W}^*.$$

Therefore,

$$\mathcal{T}\mathcal{T}^* = \mathcal{W}\mathcal{S}\mathcal{W}^*\mathcal{W}\mathcal{S}^*\mathcal{W}^*$$

$$= \mathcal{W}\mathcal{S}\mathcal{S}^*\mathcal{W}^* \tag{39}$$

and

$$\mathcal{T}^*P\mathcal{T} = \mathcal{W}\mathcal{S}^*\mathcal{W}^*P\mathcal{W}\mathcal{S}\mathcal{W}^*$$

$$= \mathcal{W}\mathcal{S}^*(\mathcal{W}^*P\mathcal{W})\mathcal{S}\mathcal{W}^* \tag{40}$$

From (39) and (40) it follows that,

$$\mathcal{W}\mathcal{S}\mathcal{S}^*\mathcal{W}^* = \mathcal{W}\mathcal{S}^*(\mathcal{W}^*P\mathcal{W})\mathcal{S}\mathcal{W}^* \tag{41}$$

Now pre-multiplying (41) by $\mathcal{W}^*$ and post-multiplying by $\mathcal{W}$, we have

$$\mathcal{S}\mathcal{S}^* = \mathcal{S}^*(\mathcal{W}^*P\mathcal{W})\mathcal{S}, \text{ where } \mathcal{W}^*P\mathcal{W} \geq 0$$

Hence, $\mathcal{S}$ is posinormal as desired. ■

Theorem 5.3.2. Let $\mathcal{T} \in \mathfrak{B}(H)$ be posinormal and $\mathcal{S}$ be an operator such that $\mathcal{T} = \mathcal{W}^*\mathcal{S}\mathcal{W}$ where, $\mathcal{W}$ is a co-isometry. Then $\mathcal{S}$ is also posinormal.

Proof: Since $\mathcal{T}$ is posinormal, we have $\mathcal{T}\mathcal{T}^* = \mathcal{T}^*P\mathcal{T}$, where P is an interrupter.

From $\mathcal{T} = \mathcal{W}^*\mathcal{S}\mathcal{W}$, it follows that $\quad \mathcal{T}^* = (\mathcal{W}^*\mathcal{S}\mathcal{W})^*$

$$= \mathcal{W}^*\mathcal{S}^*\mathcal{W}.$$

Therefore,

$$\mathcal{T}\mathcal{T}^* = \mathcal{W}^*\mathcal{S}\mathcal{W}\mathcal{W}^*\mathcal{S}^*\mathcal{W}$$

$$= \mathcal{W}^*\mathcal{S}\mathcal{S}^*\mathcal{W} \tag{42}$$

and

$$\mathcal{T}^*P\mathcal{T} = \mathcal{W}^*\mathcal{S}^*\mathcal{W}P\mathcal{W}^*\mathcal{S}\mathcal{W}$$

$$= \mathcal{W}^*\mathcal{S}^*(\mathcal{W}P\mathcal{W}^*)\mathcal{S}\mathcal{W} \tag{43}$$

From (42) and (43), it follows that,

$$\mathcal{W}^*\mathcal{S}\mathcal{S}^*\mathcal{W} = \mathcal{V}^*\mathcal{S}^*(\mathcal{W}P\mathcal{W}^*)\mathcal{S}\mathcal{W} \tag{44}$$

Now pre-multiplying (44) by $\mathcal{W}$ and post-multiplying by $\mathcal{W}^*$ we have

$\mathcal{S}\mathcal{S}^* = \mathcal{S}^*(\mathcal{W}^*P\mathcal{W})\mathcal{S}$, but $\mathcal{W}^*P\mathcal{W} \geq 0$ .This establishes that $\mathcal{S}$ is posinormal. ■

Remark 5.3.3. The following corollary is immediate from Theorem 5.3.1 and Theorem 5.3.2 above.

Corollary 5.3.4. If for any two operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ such that $\mathcal{T} = U^*\mathcal{S}U$ or $\mathcal{T} = U\mathcal{S}U^*$, where U is unitary, then $\mathcal{T}$ is posinormal whenever $\mathcal{S}$ is.

Proof: It easily follows from the proofs of Theorem 5.3.1 and Theorem 5.3.2 above since every unitary operator is either isometry or co-isometry. ■

Remark 5.3.5. The corollary above shows that posinormal operators are unitarily invariant.

5.4 Almost similarity property on subclasses of posinormal operators

Recall that, according to Jibril (1996), two operators $A, B \in \mathfrak{B}(H)$ are almost similar if the following two conditions hold;

(i) $A^*A = \mathcal{N}^{-1}(B^*B)\mathcal{N}$ and

(ii) $A^* + A = \mathcal{N}^{-1}(B^* + B)\mathcal{N}$, where $\mathcal{N}$ is an invertible operator. This property has been studied on various subclasses of posinormal operators as outlined in the following theorems:

Proposition 5.4.1. (Nzimbi *et al*., 2008). If two operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ are such that $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ and $\mathcal{S}$ is a projection, then so is $\mathcal{T}$.

Proof: For $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$, there exist an invertible operator $\mathcal{N}$, such that

$$\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})\mathcal{N} \tag{45}$$

and

$$\mathcal{S}^* + \mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\mathcal{N} \tag{46}$$

$\mathcal{S}$ is hermitian, $(\mathcal{S}^* = \mathcal{S})$ since it is a projection.

Thus, it follows from proposition 4.1.3, that $\mathcal{T}$ is also hermitian. Thus, from (45) and (46) above, we get $\mathcal{S}^2 = \mathcal{S} = \mathcal{N}^{-1}\mathcal{T}^2\mathcal{N}$ and $2\mathcal{S} = \mathcal{N}^{-1}2\mathcal{T}\mathcal{N}$ respectively.

Thus, $\mathcal{S} = \mathcal{N}^{-1}\mathcal{T}\mathcal{N}$ showing that $\mathcal{N}^{-1}\mathcal{T}^2\mathcal{N} = \mathcal{N}^{-1}\mathcal{T}\mathcal{N}$. Hence $\mathcal{T}$ is a projection. ■

Proposition 5.4.2. (Nzimbi *et al.*, 2008). If $\mathcal{S},\mathcal{T} \in \mathfrak{B}(H)$ such that $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ and $\mathcal{T}$ is hermitian, then $\mathcal{S}$ is hermitian.

Proof: Given that $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$, then there exist an invertible operator $\mathcal{N}$, such that,

$\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^*\mathcal{T})\mathcal{N}$, which implies that,

$$4\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}(4\mathcal{T}^*\mathcal{T})\mathcal{N} \tag{47}$$

Also, $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ implies that,

$\mathcal{S}^* + \mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\mathcal{N}$. It follows that,

$[\mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\mathcal{N}]\,[\mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\mathcal{N}] = (\mathcal{S} + \mathcal{S}^*)^2$. Thus,

$$\mathcal{N}^{-1}(\mathcal{T} + \mathcal{T}^*)^2\mathcal{N} = (\mathcal{S} + \mathcal{S}^*)^2 \tag{48}$$

Since T is hermitian, then we have that,

$(\mathcal{T} + \mathcal{T}^*)^2 = (2\mathcal{T})^2 = 4\mathcal{T}^2 = 4\mathcal{T}^*\mathcal{T}$. Substituting this in equation (48) above, it yields;

$$\mathcal{N}^{-1}(4\mathcal{T}^*\mathcal{T})\mathcal{N} = (\mathcal{S} + \mathcal{S}^*)^2 \tag{49}$$

Thus, by inspection on equation (47) and (48), it is easily deduced that

$4\mathcal{S}^*\mathcal{S} = (\mathcal{S} + \mathcal{S}^*)^2$.

This establishes that $\mathcal{S}$ is hermitian. ■

Proposition 5.4.3. (Nzimbi *et al.*, 2008). If $\mathcal{S},\mathcal{T} \in \mathfrak{B}(H)$ such that $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ and if $\mathcal{S}$ is hermitian, then $\mathcal{S}$ and $\mathcal{T}$ are unitarily equivalent.

Poof: Since $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ then, there exist an invertible operator $\mathcal{N}$ such that

$\mathcal{S}^* + \mathcal{S} = \mathcal{N}^{-1}(\mathcal{T}^* + \mathcal{T})\mathcal{N}$. Since $\mathcal{S}$ is normal, then they are unitarily equivalent. ■

This establishes condition for which almost similarity of operators implies similarity.

Remark 5.4.4. We note from Proposition 5.4.3 above that, such operators $\mathcal{S}$ and $\mathcal{T}$ have equal spectrum.

Proposition 5.4.5. (Nzimbi *et al.*, 2008). If an operator $\mathcal{T} \in \mathfrak{B}(H)$ is normal, then $\mathcal{T} \overset{a.s}{\sim} \mathcal{T}^*$.

Proof: If $\mathcal{T}$ is normal, there exist a unitary operator U, such that, $\mathcal{T}^* = U\mathcal{T}$ and thus, $\mathcal{T} = \mathcal{T}^* U^*$. It then follows that;

$\mathcal{T}^*\mathcal{T} = U(\mathcal{T}U^*)U^*$ and

$\mathcal{T}^* + \mathcal{T} = U\mathcal{T} + \mathcal{T}^*U^* = U\mathcal{T}^*U^* + U\mathcal{T}U^* = U(\mathcal{T} + \mathcal{T}^*)U^*$.

Thus $\mathcal{T}$ is almost unitarily equivalent and hence, almost similar to $\mathcal{T}^*$. ■

Theorem 5.4.6. Let $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$. If $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ and $\mathcal{S}$ is posinormal, then $\mathcal{T}$ is also posinormal.

Proof: Since $\mathcal{S}$ is posinormal, it implies that, $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*P\mathcal{S}$, where P is an interrupter. Also, since $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$, then there exist an invertible operator $\mathcal{N}$ such that $\mathcal{T}^*\mathcal{T} = \mathcal{N}^{-1}\mathcal{S}^*\mathcal{S}\mathcal{N}$ and

$\mathcal{T}^* + \mathcal{T} = \mathcal{N}^{-1}(\mathcal{S}^* + \mathcal{S})\mathcal{N}$.

Assuming $\mathcal{S}$ is an isometry, then from $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*P\mathcal{S}$, we have $\mathcal{S} = \mathcal{S}^*P\mathcal{S}\mathcal{S}$ and therefore, $\mathcal{S}^* = (\mathcal{S}^*P\mathcal{S}\mathcal{S})^* = \mathcal{S}^*\mathcal{S}^*P^*\mathcal{S}$.

Hence,

$$\begin{aligned} \mathcal{T}^*\mathcal{T} &= \mathcal{N}^{-1}\mathcal{S}^*\mathcal{S}^*P^*\mathcal{S}\mathcal{S}^*P\mathcal{S}\mathcal{S}\mathcal{N} \\ &= \mathcal{N}^{-1}\mathcal{S}^*\mathcal{S}^*P^*P\mathcal{S}\mathcal{S}\mathcal{N} \\ &= \mathcal{N}^{-1}\mathcal{S}^*\mathcal{S}^*\mathcal{S}\mathcal{S}\mathcal{N} \end{aligned}$$

$$= \mathcal{N}^{-1}\mathcal{S}^*\mathcal{S}\mathcal{N} \quad \text{and}$$

$$\mathcal{T}^* + \mathcal{T} = \mathcal{N}^{-1}(\mathcal{S}^*\mathcal{S}^*P^*\mathcal{S} + \mathcal{S}^*P\mathcal{S}\mathcal{S})\mathcal{N}.$$

$$= \mathcal{N}^{-1}(\mathcal{S}^*\mathcal{S}^*\mathcal{S}P^* + P\mathcal{S}^*\mathcal{S}\mathcal{S})\mathcal{N}.$$

$$= \mathcal{N}^{-1}\mathcal{S}^*\mathcal{S}(\mathcal{S}^*P^* + P\mathcal{S})\mathcal{N}.$$

$$= \mathcal{N}^{-1}(\mathcal{S}^*P^* + P\mathcal{S})\mathcal{N}, \text{ but } P \geq 0, \text{ thus we have}$$

$$= \mathcal{N}^{-1}(\mathcal{S}^* + \mathcal{S})\mathcal{N},$$

Since the posinormality of $\mathcal{S}$ justifies the almost similarity property with $\mathcal{T}$ and vice versa, then $\mathcal{T}$ is posinormal. Hence, any posinormal operators which are similar and unitarily equivalent are also almost similar. ■

Theorem 5.4.7. Let $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ be almost similar operators with their polar decompositions as $\mathcal{S} = U|\mathcal{S}|$ and $\mathcal{T} = \mathcal{V}|\mathcal{T}|$ where U and $\mathcal{V}$ are unitary. Then $\mathcal{S}$ is invertible implies $\mathcal{T}$ is also invertible.

Proof: Since $\mathcal{S} \overset{a.s}{\approx} \mathcal{T}$, then an invertible operator $\mathcal{N}$ exists such that $\mathcal{S}^*\mathcal{S} = \mathcal{N}^{-1}\,\mathcal{T}^*\mathcal{T}\mathcal{N}$ and

$\mathcal{S}^* + \mathcal{S} = \mathcal{N}^{-1}\,(\mathcal{T}^* + \mathcal{T})\mathcal{N}$. Now $\mathcal{S}$ is invertible implies that $\mathcal{S}^*$ is also invertible which also implies that $\mathcal{S}^*\mathcal{S}$ is invertible. But $\mathcal{S}^*\mathcal{S}$ is similar to $\mathcal{T}^*\mathcal{T}$ which implies that $\sigma(\mathcal{S}^*\mathcal{S}) = \sigma(\mathcal{T}^*\mathcal{T})$. Thus $\mathcal{T}^*\mathcal{T}$ is an invertible positive operator which implies that $\sqrt{\mathcal{T}^*\mathcal{T}} = |\mathcal{T}|$ is also invertible. Hence $\mathcal{T}$ is invertible. ■

Remark 5.4.8. We note that the class of invertible operators is contained in the class of posinormal operators as Theorem 5.2.11 shows.

5.5 Spectral properties of posinormal operators.

In this subsection, the spectral properties of posinormal operators are investigated. Some general results on this area are outlined below:

Lemma 5.5.1. (Asamba, 2016). Let $\mathcal{S} \in \mathfrak{B}(H)$ be posinormal operator. Supposing $\mu \in \sigma_p(\mathcal{S})$, for $0 < p < \frac{1}{2}$, then $\bar{\mu} \in \sigma_p(\mathcal{S}^*)$.

Proof: If $0 \in \sigma_p(\mathcal{S})$, then there exist a vector $\mathcal{y} \in H$, where $\mathcal{y} \neq 0$ such that $\mathcal{S}\mathcal{y} = 0$. Following that $\|\mathcal{S}\|^2 \mathcal{y} = \mathcal{S}^*\mathcal{S}\mathcal{y} = 0$ and $\|\mathcal{S}\| \geq 0$, then we have $(\mathcal{S}^*\mathcal{S})^{\frac{1}{2}k}\mathcal{y} = 0, (k = 1,2, \dots)$. For $x \in \mathbb{N}$, such that, $\frac{1}{x} < \rho$. It follows that $(\mathcal{S}^*\mathcal{S})^{\frac{1}{2}x}\mathcal{y} = 0$ and $(\mathcal{S}^*\mathcal{S})^{\frac{1}{2}\rho}\mathcal{y} = 0$. Since $\mathcal{S}$ is posinormal, then we have, $(\mathcal{S}^*\mathcal{S})^{\rho}\mathcal{y} = 0$. Hence, $\mathcal{S}^*\mathcal{S}\mathcal{y} = 0$. Next, supposing that, $\mu \in \sigma_p(\mathcal{S})$, for $\mu \in \mathbb{C}$, where $\mu \neq 0$. It therefore follows that if exist $m \in H$ such that $m \neq 0$ then we have, $\mathcal{S}m = \mu m$. If we let $\mathcal{S} = U\|\mathcal{S}\|$ be a polar decomposition of $\mathcal{S}$ and U being a unitary operator, for $\|\mathcal{S}\|m = \mu m$, it follows that

$\|\mathcal{S}\|^{\frac{1}{2}}U\|\mathcal{S}\|^{\frac{1}{2}}\|\mathcal{S}\|^{\frac{1}{2}}m = \mu\|\mathcal{S}\|^{\frac{1}{2}}m$. We know that $\tilde{\mathcal{S}} = \|\mathcal{S}\|^{\frac{1}{2}}U\|\mathcal{S}\|^{\frac{1}{2}}$ and consequently,

$\tilde{\mathcal{S}}^* = \|\mathcal{S}\|^{\frac{1}{2}}U^*\|\mathcal{S}\|^{\frac{1}{2}}m = \bar{\mu}\|\mathcal{S}\|^{\frac{1}{2}}m.$

Therefore, $\mathcal{S}^* = (\|\mathcal{S}\|m) = \bar{\mu}\|\mathcal{S}\|m$. Since $\|\mathcal{S}\|m \neq 0$, then $\bar{\mu} \in \sigma_p(\mathcal{S}^*)$. ■

Theorem 5.5.2. (Asamba,2016). Let $\mathcal{S} \in \mathfrak{B}(H)$ be posinormal operator. Then $\sigma(\mathcal{S}) = \{\mu : \bar{\mu} \in \sigma_\pi(\mathcal{S}^*)\}$.

Proof: For $\sigma(\mathcal{S}) = \sigma_\pi(\mathcal{S}) \cup \{\mu: \bar{\mu} \in \sigma_\pi(\mathcal{S}^*)\}$, it suffices to show that

$\sigma(\mathcal{S}) = \{\mu: \bar{\mu} \in \sigma_\pi(\mathcal{S}^*)\}$. Supposing that, $\mu \in \sigma_\pi(\mathcal{S})$, then we have $\mu \in \sigma_p(\mathcal{T}(\mathcal{S})$, where $\mathcal{T}$ is a mapping. It follows that $\bar{\mu} \in \sigma_p(T(S^*))$ since $\mathcal{T}(\mathcal{S})$ is posinormal. Since $\sigma_p\big(\mathcal{T}(\mathcal{S}^*)\big) = \sigma_\pi(\mathcal{S}^*)$, it follows that $\bar{\mu} \in \sigma_\pi(\mathcal{S}^*)$. ∎

Lemma 5.5.3. (Asamba, 2016)**.** Let $\mathbb{S} = (\mathcal{S}_1, \dots \mathcal{S}_n)$ be doubly commuting n −tuple of posinormal operator on H. Supposing $\mu = (\mu_1, \dots \mu_n) \in \sigma_p(\mathbb{S})$, then

$\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_n) \in \sigma_\pi(\mathbb{S}^*)$, where $\mathbb{S}^* = \mathcal{S}_1^*, \dots, \mathcal{S}_n^*$.

Proof: For a vector $y \in H$, where $y \neq 0$, such that $\mathcal{S}_i y = \mu_i y$ $(i = 1, \dots, n)$, We therefore assume that $\mu_i, \dots \mu_k$ are zero norm and μ_{k+1}=…= $\mu_n = 0$. Thus

$\mathcal{S}_{k+1}^* = \mathcal{S}_y^* = 0$ is obtained. Again, $\mathcal{S}_1^*(\|\mathcal{S}\| y) = \bar{\mu}_1 \|\mathcal{S}_i\| y$, where $\mathcal{S}_i > 0$ in a polar decomposition $\mathcal{S}_i = U_i \|\mathcal{S}_i\|$, where $i = 1, \dots, k$.

If $\|\mathcal{S}_1\| \dots \|\mathcal{S}_k\| y = 0$ and since $(\mathcal{S}_1, \dots, \mathcal{S}_k)$ is doubly commuting k-tuple of posinormal operators, then U_i and $\|\mathcal{S}_i\|$commute with U_j and$\|\mathcal{S}_j\|$, for every $i \neq j$. Therefore, we have $\mathcal{S}_1\mathcal{S}_2 \dots \mathcal{S}_k y = 0$ and it follows that $\mu_1, \dots \mu_k = 0$, since every $\mu_i \neq 0$, where $i = 1, \dots, k$.

Hence, $\|\mathcal{S}_1\| \dots \|\mathcal{S}_k\| y \neq 0$, and

$$\text{consequently } \mathcal{S}_i^*(\|\mathcal{S}_1\| \dots \|\mathcal{S}_k\| y) = \|\mathcal{S}_1\| \dots \|\mathcal{S}_{i-1}\| \|\mathcal{S}_{i+1}\| \dots \|\mathcal{S}_k\| \|\mathcal{S}_i^*\| \|\mathcal{S}_i\| y$$
$$= \bar{\mu}_1(\|\mathcal{S}_1\| \dots \|\mathcal{S}_k\| y.$$

We again have the following since $\mathcal{S}_i$ also commute with, $\|\mathcal{S}_1\| \dots \|\mathcal{S}_k\|$;

$\mathcal{S}_i^*(\|\mathcal{S}_1\| \dots \|\mathcal{S}_k\| y) = 0, i = k+1, \dots, n$. Hence, $\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_n) \in \sigma_p(\mathbb{S}^*)$. ∎

Theorem 5.5.4. (Asamba, 2016). Let $\mathbb{S} = (\mathcal{S}_1, \dots \mathcal{S}_n)$ be doubly commuting n-tuple of posinormal operators on H. Then $\sigma(\mathbb{S}) = \{(\mu_1, \dots \mu_n) \in \mathbb{C}^n: (\bar{\mu}_1, \dots, \bar{\mu}_n) \in \sigma_\pi(\mathbb{S}^*)\}$.

Proof: Given $\mathbb{S}$ is a doubly commuting n-tuple, we have $(\mu_1, \dots \mu_n) \in \sigma(\mathbb{S})$. If there exists some partition $\{i_1, \dots, i_m\} \cup \{j_1, \dots, j_s\} = \{1, \dots, n\}$ and also a sequence $\{\mathcal{y}_k\}$ of vectors in H such that $(\mathcal{S}_{i\tau} - \mu_{i\tau})y_k \to 0$ and $(\mathcal{S}_{jv} - Z_{jv})^* y_k \to 0$ as $k \to \infty$, for

$\tau = 1, \dots, m$ and $v = 1, \dots, s$, then for a mapping $\mathcal{T}$ such that $(\mathcal{S}_{i1}, \dots \mu_{jm}, \bar{\mu}_{j1}, \dots, \bar{\mu}_{js}) \in \sigma_\pi(\mathcal{T}(\mathcal{A})$, we thus have $\mathcal{T}(\mathcal{A}) = (\mathcal{T}(\mathcal{S}_{i1}, \dots, \mathcal{T}\left(\mathcal{S}_{ji}^*\right)$.

Since $\mathcal{T}(\mathcal{S}_i)$ is a posinormal operator, it follows that $(\bar{\mu}_1, \dots, \bar{\mu}_n) \in \sigma_p(\mathcal{T}(\mathcal{S}^*))$. Hence,

$(\bar{\mu}_1, \dots, \bar{\mu}_n) \in \sigma_\pi(\mathcal{S}^*)$. Thus it is clear that, $\sigma_\pi(\mathcal{S}^*) \subset \sigma(\mathcal{S})$ and so

$\sigma(\mathbb{S}) = \{(\mu_1, \dots \mu_n) \in \mathbb{C}^n : (\bar{\mu}_1, \dots, \bar{\mu}_n) \in \sigma_\pi(\mathbb{S}^*)\}$. ■

Corollary 5.5.5. (Rhaly, 1994). Supposing $\mathcal{S} \in \mathfrak{B}(H)$ and $\lambda \notin \sigma(\mathcal{S})$, then $\mathcal{S} - \lambda$ is posinormal.

Proposition 5.5.6. (Rhaly, 2016). The following properties hold true for posispectrum, $\sigma_{po}(\mathcal{S})$:

(i) $\mathcal{S}$ is dominant if and only if $\sigma_{po}(\mathcal{S}) = \emptyset$, i.e. if $\mathcal{S}$ is normal or hyponormal, then $\sigma_{po}(\mathcal{S}) = \emptyset$.

(ii) $\mathcal{S}$ is posinormal if and only if $0 \notin \sigma_{po}(\mathcal{S})$.

(iii) $\pi_1(\mathcal{S}) \subseteq \sigma_{po}(\mathcal{S}) \subseteq \sigma(\mathcal{S})$.

(iv) There exists operators for which the posispectrum is topologically large in that it may contain a nonempty open set.

$(v) \sigma_{po}(\mathcal{S}) = \sigma_{po}(\mathcal{S}^*)^*$

$$\Leftrightarrow \left\{Ran\,(\lambda I - \mathcal{S}) \subseteq Ran\left(\bar{\lambda} I - \mathcal{S}^*\right) \Leftrightarrow Ran\left(\bar{\lambda} I - \mathcal{S}^*\right) \subseteq Ran\,(\lambda I - \mathcal{S})\right\}$$

$\forall\, \lambda \in \mathbb{C}$. Also,

$$Ran\,(\lambda I - \mathcal{S}) \subseteq Ran\left(\bar{\lambda} I - \mathcal{S}^*\right) \Leftrightarrow Ran\left(\bar{\lambda} I - \mathcal{S}^*\right) = Ran\,(\lambda I - \mathcal{S}), \forall \lambda \in \mathbb{C}.$$

$(vi)\ \sigma_{po}(\mathcal{S}) = \sigma_{po}(\mathcal{S}^*) = \emptyset \Leftrightarrow Ran\big(\bar{\lambda}I - \mathcal{S}^*\big) = Ran\ (\lambda I - \mathcal{S})$, for every $\lambda \in \mathbb{C}$.

Thus $\mathcal{S}$ is dominant and codominant.

Proof: Properties (i) and (ii) are direct. Therefore, by range inclusion criterion we obtain;

$\sigma_{po}(\mathcal{S}) = \{\lambda \in \mathbb{C}: \lambda I - \mathcal{S}$ is not posinormal $\}$.

$= \{\lambda \in \mathbb{C}: Ran\ (\lambda I - \mathcal{S}) \nsubseteq Ran\big(\bar{\lambda}I - \mathcal{S}^*\big)\}$.

Since $\mathcal{S}$ is dominant if and only if $\lambda I - \mathcal{S}$ is posinormal, $\lambda \in \mathbb{C}$. This shows by the range inclusion criterion that,

$Ran\ (\lambda I - \mathcal{S}) \subseteq Ran\big(\bar{\lambda}I - \mathcal{S}^*\big), \forall\ \lambda \in \mathbb{C}$. This shows that $\mathcal{S}$ is dominant $\Leftrightarrow \sigma_{po}(\mathcal{S}) = \emptyset$ and

$\mathcal{S}$ is posinormal $\Leftrightarrow 0 \notin \sigma_{po}(\mathcal{S})$. Particularly, if $\mathcal{S}$ is normal or hyponormal, then $\sigma_{po}(\mathcal{S}) = \emptyset$.

For (iii), Since every invertible operator is posinormal and since $\lambda \in \mathbb{C}$ for which $\lambda I - \mathcal{S}$ is invertible lies in the complement of the spectrum it thus follows that $\sigma_{po}(\mathcal{S}) = \sigma(\mathcal{S})$.

It is also notable that

$\pi_0(\mathcal{S}) = \big\{\lambda \in \mathbb{C}: Ker(\lambda I - \mathcal{S}) = \{0\}\big\} = \big\{\lambda \in \mathbb{C} = Ran\big(\bar{\lambda}I - \mathcal{S}^*\big)^- \neq\ H\big\}$, and

$\pi_1(\mathcal{S}) = \{\lambda \in \pi_0(\mathcal{S}): Ran(\lambda I - \mathcal{S}) = H\}$

$= \big\{\lambda \in \mathbb{C}: Ran\big(\bar{\lambda}I - \mathcal{S}^*\big)^- \subset Ran\ (\lambda I - \mathcal{S}) = H\big\}$.

Since $Ran\ (\mathcal{S})$ is closed if and only if $Ran\ (\mathcal{S}^*)$ is closed for every $\mathcal{S} \in \mathfrak{B}(H)$, then the above proper inclusion can be rewritten as $\pi_1(\mathcal{S}) = \{\lambda \in \mathbb{C}: Ran(\bar{\lambda}I - \mathcal{S}^*) \subset Ran\ (\lambda I - \mathcal{S}) = H\}$. Thus, $\pi_1(\mathcal{S}) \subseteq\ \sigma_{po}(\mathcal{S})$ as required.

For (iv), since it is known fact that $\pi_1(\mathcal{S})$ is always open subset of $\mathbb{C}$ and also that there are Hilbert space operators $\mathcal{S}$ for which $\pi_1(\mathcal{S})$ is nonempty , therefore, (iv) follows from (iii).

For (v), considering the set $\sigma_{po}(\mathcal{S}^*)^* = \{\lambda \in \mathbb{C}: Ran(\bar{\lambda}I - \mathcal{S}^*)Ran\ (\lambda I - \mathcal{S})\}$ and taking $\rho_{po}(\mathcal{S}) = \mathbb{C}\backslash\sigma_{po}(\mathcal{S}) = \{\lambda \in \mathbb{C}: Ran(\lambda I - \mathcal{S}) \subseteq Ran(\bar{\lambda}I - \mathcal{S}^*)\}$,
$\rho_{po}(\mathcal{S}^*)^* = \mathbb{C}\backslash\sigma_{po}(\mathcal{S}^*)^* = \{\lambda \in \mathbb{C}: Ran(\bar{\lambda}I - \mathcal{S}^*) \subseteq Ran\ (\lambda I - \mathcal{S})\}$. Therefore,
$\sigma_{po}(\mathcal{S}) = \sigma_{po}(\mathcal{S}^*)^* \Leftrightarrow \rho_{po}(\mathcal{S}) = \rho_{po}(\mathcal{S}^*)^* \Leftrightarrow \{Ran(\lambda I - \mathcal{S}) \subseteq Ran(\bar{\lambda}I - \mathcal{S}^*)Ran(\bar{\lambda}I - \mathcal{S}^*) \subseteq Ran\ (\lambda I - \mathcal{S})\ \forall\ \lambda \in \mathbb{C}$. This establishes the proof.

Finally for (vi), it follows from (i) since $\sigma_{po}(\mathcal{S}) = \sigma_{po}(\mathcal{S}^*) = \emptyset$ if and only if both $\mathcal{S}$ and $\mathcal{S}^*$ are dominant. ■

CHAPTER SIX

SUMMARY AND RECOMMENDATIONS

6.1 Summary

This research studied *almost similarity* property considerably on partial isometries, θ-operators and posinormal operators, where chapters one and two are devoted to introduction and literature review respectively.

In chapter three, we have exhibited a number of results on the property of almost similarity involving subclasses of partial isometries see (Theorem 3.2.3, Corollary 3.3.2, Proposition 3.4.1, 3.4.2 and 3.4.5).

We further showed that for any two operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$, such that $\mathcal{S} \overset{a.s}{\sim} \mathcal{T}$ if $\mathcal{S}^2$ is a partial isometry and $\mathcal{T}$ is self-adjoint, then $\mathcal{T}^2$ is also partially isometric, see Theorem 3.4.9.

In chapter four, it is shown that for a θ-operator $\mathcal{T} \in \mathfrak{B}(H)$ and either $\mathcal{S} = U\mathcal{T}U^*$, with U isometry or $\mathcal{S} = U^*\mathcal{T}U$, with U a co-isometry then $\mathcal{S}$ is also a θ-operator, see (Theorems 4.3.1 and 4.3.2). In Theorem 4.3.5, it is shows that if two operators are unitarily equivalent and one of them is a θ-operator, then so is the other.

We have also shown through Proposition 4.4.4 that if two operators $\mathcal{T}$ and $\mathcal{S}$ are almost similar and $\mathcal{S}$ is a θ-operator, then so is $\mathcal{T}$. On this class of θ-operators, we also extended the property of almost similarity to that of $\propto$-almost similarity. To this end we showed that if $\mathcal{T}$ and $\mathcal{S}$ are projections which are $\propto$-almost similar, then under further condition we get that they have not only equal spectra but also equal approximate point spectrum, see (Proposition 4.5.6).

Lastly, Corollary 4.5.10 shows that for two operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ and $\mathcal{N}$ be an invertible operator such that $\mathcal{N}\mathcal{S} = \mathcal{T}\mathcal{N}$ where $\mathcal{S}$ and $\mathcal{T}$ satisfy the Pantum-Fuglede property. Then $\mathcal{S}$ and $\mathcal{T}$ are α −almost similar.

In chapter five, it is shown that for two operators $\mathcal{S}, \mathcal{T} \in \mathfrak{B}(H)$ such that $\mathcal{S}$ is a posinormal and either $\mathcal{S} = U\mathcal{T}U^*$ or $\mathcal{S} = U^*\mathcal{T}U$, where U is unitary, then $\mathcal{T}$ is also posinormal, see (Theorem 5.3.1 and theorem 5.3.2). Thus, $\mathcal{S}$ and $\mathcal{T}$ are unitarily equivalent posinormal operators.

Therefore, the classes of θ-operators and posinormal operators are not only unitarily invariant but also isometrically and co-isometrically invariant. Also, with respect to almost similarity of operators, if two operators are similar with one of them a θ-operator, then the other is also a θ-operator. Also, Lemma 5.5.1, Theorem 5.5.2, Theorem 5.5.4 and Proposition 5.5.6 give essential information about spectral properties of operators.

6.2 Recommendations

The following are recommended for further studies;

(i) In both chapter three and four we have managed to show that if two operators are almost similar and one of them is either partial isometry or θ- operator then so is the other operator. However, in chapter five we have not succeeded in showing the same result for posinormal operators. This is an area for further investigation.

(ii) In all the three chapters (3, 4 & 5), we have only shown that equality of spectra is realized when the two given operators are unitarily equivalent or similar but not when they are almost similar .This can also be investigated further.

(iii) In this thesis we have studied the property of almost similarity on three different classes of operators namely *Partial isometries*, θ- *operators* and *Posinormal operators*. The same study can be undertaken on different sets of classes of operators.

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