STUDY OF STEADY AND UNSTEADY VISCOUS INCOMPRESSIBLE MHD FLUID FLOW

BY

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DECLARATION

Declaration by the Candidate

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DEDICATION

To my beloved wife Veronicah N. Wambua, the love of my wonder and my daughter Trizah M. Wambua.

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ABSTRACT

An analytical study of laminar steady, viscous, incompressible Couette fluid flow between two infinite parallel plates under the influence of transverse magnetic field is studied. The resulting governing partial differential equation was solved analytically by Sumudu Transform for the linear differential equation with constant coefficients. The Couette flow velocity profiles for various Hartmann number and various angles of inclinations were presented graphically. The results showed that, increase in magnetic field strength and magnetic inclinations resulted into decrease in velocity profiles. The motion of two dimensional steady laminar Poiseuille flow of a viscous MHD incompressible fluid between two parallel porous plates under the influence of uniform transverse magnetic field was also examined. The resulting coupled differential equations were solved numerically using finite difference approach. The numerical computation of the generated linearized system of equations was achieved with the aid of MATLAB application software. The results depicted graphically showed that, an increase in Hartmann number led to decrease in Poiseuille flow velocity distribution which was as a result of Lorentz force which offered resistance opposing the fluid motion. Unsteady MHD Couette laminar flow of viscous incompressible fluid between two parallel porous plates in presence of uniform magnetic field was also investigated. The upper and lower plates were maintained at two different but constant temperatures. A sudden uniform and a constant pressure gradient, an external uniform magnetic field was applied in the positive y - direction. The flow was subjected to a uniform suction from above and uniform injection from below at $t \geq 0$. The resulting linear differential equations were solved numerically using finite difference approach. The Crank-Nicolson implicit method was used at two successive time levels so as to determine the velocity and temperature distributions for different values of the parameters M ,S and α. The results showed that, when suction was suppressed, increasing the porosity parameter had no marked effect on velocity distribution but increasing the suction resulted into a decrease in the velocity which reached the steady state monotonically with time due to convection of the fluid from regions in the lower half of the centre of channel. It was also reported that, increasing suction decreased the temperature at the centre of the channel for all values of t due to influence of convection in the pumping of the fluid from the lower cold region of the channel towards the centre.

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LIST OF ABBREVIATIONS, ACRONYMS AND SYMBOLS

ϵ	Strain rate
ϵ_o	Capacitivity
μo	Vacuum permeability
σ	Electrical conductivity
ρ	Fluid density
θ	shear strain
B	Magnetic flux density also called magnetic induction or magnetic field
Bo	Characteristic Magnetic field strength used to nondimensionalize equations
C_p	Specific heat at constant pressure
D	Electric displacement field also called electric flux density
da	Differential element of surface area with infinitesimally small magnitude and direction normal to surface S
dl	Differential element of path length tangential to contour C enclosing surface S
dV	Differential element of volume V enclosed by surface S
E	Electric field
E_c	Eckert number
F	Force
F_i	Inertial force
F_v	Viscous force
F_X	Component of force in x- direction
F_Y	Component of force in y- direction
F_Z	Component of force in z- direction
Gr	Grashof number
H	Magnetic field intensity also called auxiliary field
Ha	Hartman number
I	Electric current

J	free current density, not including polarization or magnetization currents bound in a material
k	Thermal conductivity
K	Darcy permeability
M	Porosity parameter
Nu	Nusselt number
p	Fluid pressure
Pr	Prandtl number
Q	Electric charge
r	Distance between two charges
Re	Reynolds number
S	Suction/injection parameter
ua	Average velocity
Vg	Velocity gradient

OPERATIONAL DEFINATION OF TERMS

i. Fluid

This is that state of matter that is capable of flowing and changing shape. Fluid can be perfect fluid or real fluid.

Perfect fluid is a fluid which is incapable of exerting shearing stress whether at rest or in motion and the pressure that it exerts in any surface is always along the normal to the surface at that point. It is a fluid which is *inviscid* and whose density is a constant. Since there is absence of shear stresses, then the adjacent layers of an ideal fluid can move at two different velocities (slip flow) without affecting each other by internal frictional forces. The only influence they exert on each other is through their geometry, which must conform to the geometry of the solid boundary.

Real fluid is fluid which has *viscosity* and unlike perfect fluids, real fluids cannot slip with a finite velocity difference over adjacent layers or over solid boundaries. Thus, a *viscous fluid* exerts shearing stress and offer resistance to the body moving through it as well as between the particles of the fluid itself. The amount of viscosity that governs the stickiness of fluid layers will be responsible for making gradual the velocity variation across the layers. Near a stationary boundary, the velocity of a real fluid must gradually increase from zero at the boundary to a finite stream velocity in a finite fluid layer generally called Prandtl's layer.

ii. Rate of Shear Strain

This is the rate of change of the mainstream velocity in the transverse direction. It is given as

$$\epsilon = \frac{d\theta}{dt}$$

$$\epsilon = \lim_{\partial t \to 0} \frac{\partial \theta}{\partial t} = \frac{du}{dy}$$

where ϵ is strain rate and θ is the shear strain.

iii. Ohm's Law

Ohm's law states that, "when a steady current is flowing through a conductor, the potential difference between its ends is directly proportional to the current provided that the physical condition of the conductor does not change." Thus if V is the potential difference, and I is the current, we may write

$$V = IR \tag{0.1}$$

where R is the constant termed as the resistance of the conductor at a given temperature and this is measured in ohms.

A conductor has a resistance equal to one Ohm of resistance if the potential difference (*p.d*) between its ends is one e.m.u of potential and when the current through it is one *e.m.u* of the current.

Thus

$$one\ ohm = \frac{one\ volt}{one\ ampere}$$

$$one\ ohm = \frac{10^{8} e.m.u\ of\ potential}{10^{-1} e.m.u\ of\ current} = 10^{9} e.m.u \tag{0.2}$$

iv. Coulomb's Law

Coulomb in 1785 used a torsion balance to investigate the force exerted by one charge upon a second and discovered that this force varied inversely as the square of the distance between the charges and acted along the line joining the two charges. Further investigations demonstrated that the force is directly proportional to the product of the two charges. These can be written as

$$F \propto \frac{Q_1 Q_2}{r^2} \tag{0.3}$$

where Q_1 and Q_2 are the magnitude of the charges and r is the distance between them. In experiment of this type, force experienced by the second charge depends on the medium the charges are placed in the surrounding and, in particular, on the nature of any material in which it may be immersed. The above Coulomb's law equation refers solely to the force exerted on Q_1, Q_2 and vice-versa. If Q_1 and Q_2 are charges of the same sign, so that F *is* positive, it is found that F is a repulsive force. Thus forces exerted by Q_1 and Q_2 acts along the line joining the two charges and is in the direction from Q_1 towards Q_2 and vice-versa.

Coulomb's law is expressed as an equation if a constant of proportionality and the equation represents the force between two charges in a vacuum is introduced. The equation is given as

$$F = \frac{Q_1 Q_2}{4\pi \varepsilon_o r^2} \tag{0.4}$$

Where $\frac{1}{4\pi \varepsilon_o} = 8.9875517873681764 \times 10^9 N$ and ε_o is the permittivity of free space.

v. Gauss law

An extension of the coulomb's law is the Gauss law. Consider a charge q within a volume having surface a. Gauss law is then written as

$$\iint_a \mathrm{E} \cdot \hat{n} da = \frac{q}{\varepsilon_o} \tag{0.5}$$

If q is the electric charge enclosed by the surface a, i.e. $q = \iiint_v \rho_e \, dv$ where the charge density ρ_e is the charge per unit volume defined as $\rho_e = q(n^+ + n^-) = \sum_{i=1} n_i q_i$ where n^+ and n^- are respectively the numbers of positive and negative charges in a unit volume. Then it follows that, $\nabla \cdot \mathrm{E} = \frac{\rho_e}{\varepsilon_o}$,

where $\varepsilon_o = \frac{10^7}{4\pi c^2} \sec^2 m^2$ with $c^2 = 3 \times 10^8 m \sec^{-1}$.

CHAPTER ONE

INTRODUCTION

1.1 General introduction

Magnetohydrodynamics (MHD) is the study of motion of a highly conducting fluid such as plasma and salt water through a magnetic field. The fundamental concept behind MHD is that, magnetic field can induce currents in a moving conductive fluid, which in turn creates forces on the fluid and also changes the magnetic field itself. It involves the solution of Maxwell's equations coupled with the equation of motion and continuity equation of the fluid. The set of equations which describe MHD are a combination of Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism.

The Navier-Stokes equations are partial differential equations that determine the velocity of the fluid at any particular instant of time. The Maxwell's equations on the other hand are four partial differential equations that combine together to form complex equations either magnetic or electric field or both. When Maxwell's equations are coupled with Navier-Stokes equations, they are very useful in studying the working of MHD systems like MHD generators. These differential equations have to be solved simultaneously either analytically or numerically.

The most characteristic feature of MHD is undoubtedly the coupling between the electromagnetic and mechanical forces. Thus, the motion of the conducting fluid across the magnetic field generates electric currents which alter the magnetic fluid. Thus, the interaction of the two forces produces MHD.

In designing MHD device, the engineer must know properties and the kinematics of plasmas. Magnetofluidmechanics comes about when ionized fluid interacts with magnetic forces. Ionized gases are often called plasmas and is usually considered to be the fourth physical state of matter in addition to solid, liquid and gases. It is that state of matter where molecules are fully or partially contain enough free, charged particles for its dynamical behavior to be determined under the influence of electromagnetic forces. Since there is no definite volume in plasma, gases and plasma are both treated as gases. Magnetism is a complex subject from a mathematical point of view than electrostatics. This is because currents rather than charges are the source of magnetism, thus the current density is a vector quantity.

The interaction of conducting fluids with electric and magnetic fields provides a variety of phenomenon associated with the electromagnetic-fluid-mechanical energy conversion. Such interactions can be observed in liquids, gases, two-phase mixtures or plasmas. Many scientific and technical applications exist such as heating and flow control in metal processing, power generation from two-phase mixtures and dynamos that create magnetic fields in planetary bodies. Several terms are applied to the broad field of electromagnetic effects in conducting fluids, such as magneto-fluid-mechanics, magneto-gas-dynamics and the more common one used and which is used here is MHD.

The phenomenal growth of electromagnetic brakes, MHD power generators, flow meter and electromagnetic pump in the recent years has been attracting considerable attention all over the world. MHD flow of electrically conducting fluid between parallel plates has an important applications in MHD pumps, generators, flow meters, liquid metals MHD, and physiological fluid flow. Since the early part of the 20^{th} century, practical MHD devices have been in use. For example, an MHD pump Prototype was built as early as 1907 (Northrup, 1907). In recent times MHD devices have been used for stirring, levitating and controlling flows of liquid metals for metallurgical processing and other applications (Kolesnichenko, 1990). Gas-phase MHD is best known in MHD power generation. From 1959, major efforts have been carried around the world to develop this technology to improve electric conversion efficiency, increase reliability by eliminating moving parts and reduce emissions from coal and gas plants (Sporn & Kantrowitz, 1959; Steg and Sutton, 1960). Some novel applications are still in development. For instance, research has shown the possibility of sea water propulsion using MHD (Graneau,1989) and control of turbulent layers to reduce drag (Tsinober,1990). A lot of research and extensive worldwide on magnetic confinement of plasmas has led to attainment of conditions approaching those needed to sustain fusion reactions (Baker et al., 1998).

The basic concept describing MHD phenomena is as illustrated in Figure 1.1.

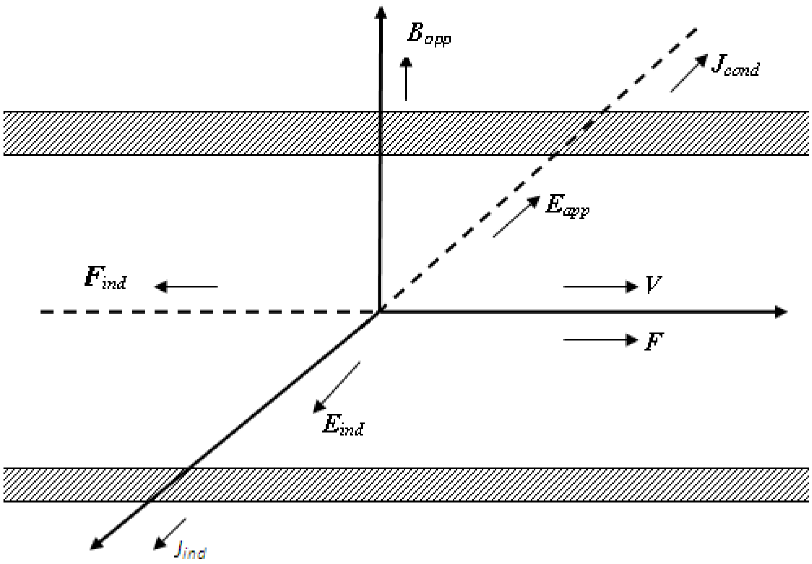

Figure 1.1 Conceptual Framework of MHD

KEY

B*app*	Applied magnetic field
J*cond*	Conducting current density due to applied electric field
E*app*	Applied electric field
E*ind*	Induced electric field
F*ind*	Induced electromotive force
V	Velocity of the conducting fluid
F	Lorentz force
J*ind*	Induced current density due to electric field induced by applied magnetic field

Consider now an electrically conducting fluid having velocity **V**. At right angle to this apply a magnetic field represented by the vector B_{app} and assume that steady flow conditions have been attained. Steady flow is a flow in which the velocity, pressure, density and other such characteristics at a point do not change with time. Steady flow specifies a limitation on time variation and not space variation. Thus, at various points of the flow field, all such quantities associated with the flow field will remain unchanged with time. In steady motion, time drops out of the independent variables and the various field quantities simply become functions of the spatial co- ordinates. For analytic reasons, the term "steady state" is required to imply that no macroscopic charge density is being built up at any place in the system, as well as all currents are constant in time. Due to the interaction of the two fields, an electric field denoted by $\mathbf{E}_{ind}$ is induced at right angles to both V **and** B_{app}. This electric field is given mathematically by $\boldsymbol{E}_{ind=\boldsymbol{V}\times\boldsymbol{B}_{app}}$.

Assuming that the conducting fluid is and remains isotropic in spite of the magnetic field, then we can denote the electrical conductivity by a scalar quantity σ. Then by $Ohm^t s$ law, the density of the current induced in the conducting fluid and denoted by $\mathbf{J}_{ind}$ is $\mathbf{J}_{ind} = \sigma\mathbf{E}_{ind}$. Simultaneously occurring with the induced current is the ponderomotive force $\mathbf{F}_{ind}$ which is given by $\mathbf{F}_{ind} = \mathbf{J}_{ind} \times \mathbf{B}_{app}$. The ponderomotive force is well known as the driving force of an electric motor. This force occurs because, as in an electric generator, the conducting fluid cuts the lines of the magnetic field. Because the vector product of $\mathbf{F} = \mathbf{J} \times \mathbf{B}$ yield a vector perpendicular to both $\mathbf{J}$ and $\mathbf{B}$, the induced force is parallel to $\mathbf{V}$ but opposite in direction. To make the configuration slightly more general, let us apply an electric field $\mathbf{E}_{app}$ at right angles to both $\mathbf{B}_{app}$ and $\mathbf{V}$, but opposite in direction to $\mathbf{J}_{ind}$. The current density due to applied electric field is denoted by $\mathbf{J}_{cond}$ and it is called conducting current. The net current density $\mathbf{J}$ through the conducting fluid and the one we would be measuring with suitably placed ammeter is then given by

$$\mathbf{J} = \sigma\,[\mathbf{E}_{app} + \mathbf{V} \times \mathbf{B}] = \sigma\,[\mathbf{E}_{app} + \mathbf{E}_{ind}] \tag{1.1}$$

where $\sigma\mathbf{E}_{app}$ is the electrostatic force. The ponderomotive or Lorentz force associated with this current is then

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} = \delta[\mathbf{E}_{app} + \mathbf{V} \times \mathbf{B}_{app}] \times \mathbf{B}_{app} \tag{1.2}$$

If $\mathbf{E}_{app} > \mathbf{V} \times \mathbf{B}_{app}$, we then have an accelerator which may be used as thrust-producing device.

The laminar flow through a channel under uniform transverse magnetic field is important because of the use of MHD generator, the MHD pump, crude oil purification and the electromagnetic flow meter. The general model that is normally considered in these studies consist of an infinitely long channel of constant cross section with a uniform static magnetic field applied transverse to the axis of the channel. The walls of the channel are insulators, conductors or a combination of insulators and conductors depending on the intended application. For example, in the MHD generator and pump, the channel cross section is normally rectangular with insulated walls perpendicular to the magnetic field. For the electromagnetic flow meter case, the channel cross section is normally circular with conducting walls.

1.2 Maxwell's field equations

When electric field and magnetic field give rise to one another, an electromagnetic field occurs this is described by Maxwell's field equations. Maxwell deduced these equations from empirical equations of *Faraday* and *Ampere*. *Faraday*, experiment- ing with induction coil observed that, when a current is passed through a coil, there will be a momentary flow of current in a coil located close by. He further noted that the induced current is proportional to the time rate of charge of the induced magnetic field. The *Faraday*^{t}*s* induction law is thus given as

$$\boldsymbol{E}.\boldsymbol{dl} = -\frac{d}{dt}\iint_a \mathbf{B}\cdot\hat{n}\, da \tag{1.3}$$

where **dl** is an infinitesimal vector length and $\hat{\mathbf{n}}$ is the outwardly directed unit normal vector on the elemental area *da*. It follows from equation (1.3) that an increasing magnetic flux through a loop gives rise to a negative electromotive force about the loop over which the line integral is written. This equation defines the magnitude of the field induced by an electric field and relates a line integral to a surface integral.

Thus by *Stoke*^{t}*s* theorem, equation (1.3) can be transformed into a surface integral as:-

$$\oint \mathrm{E}\cdot \mathrm{dl}=\iint_{\mathrm{a}}\left(\nabla\times \mathrm{E}\right)\cdot \hat{n}\, da=-\frac{\partial}{\partial t}\iint_{a}\mathrm{B}\cdot \hat{n}\ \ \mathrm{da} \tag{1.4}$$

and for some surface of integration, the surface integrals in (1.4) are equal, hence, equation (1.4) can be simplified as

$$\frac{\partial \mathrm{B}}{\partial t}+\nabla\times \mathrm{E}=0 \tag{1.5}$$

Equation (1.5) is the differential form of one of the four Maxwell's equations to be considered.

The interaction of electricity and magnetism is once manifested in equation (1.5) and in pure electrostatic problem, for time independent EM fields, then **B** does not change with time, hence equation (1.5) becomes $\nabla \times \mathbf{E} = 0$. Since curl of a vector is identically zero we deduce that, electrostatic field is irrotational.

Similarly, the divergence of the curl of any vector is zero. Taking the divergence of equation (1.5) gives

$$\nabla \cdot(\nabla \times \mathbf{E}) = 0 \tag{1.6}$$

or

$$\frac{\partial}{\partial t}(\nabla\cdot \mathrm{B})+\nabla\cdot(\nabla\times \mathrm{E})=0 \tag{1.7}$$

but $\nabla \cdot(\nabla \times \mathbf{E}) = 0$, hence equation (1.7) reduces to

$$\frac{\partial}{\partial t}\ \nabla\cdot \mathrm{B}\ \ = 0 \tag{1.8}$$

and therefore getting

$$\nabla \cdot \mathbf{B} = \mathrm{c} \tag{1.9}$$

and conclude at once that the divergence of **B** is time invariant.

Considering experimental evidence, we determine the value of constant c in equation (1.9) above. It is known that, there are no magnetic sources in the same sense that there are electric charges. In fact, by experience we cannot conceive of any situation in which the divergence of **B** has been other than zero, and thus postulate that the constant in equation (1.9) is zero. We then have

$$\nabla \cdot \mathbf{B} = 0 \tag{1.10}$$

Consider the electric displacement vector **D** and in particular the divergence of **D**. An extension of *Coulomb*^{t}*s* law is the *Gauss*^{t} law. If we consider a charge q within a volume having surface a , *Gauss*^{t} law is then written as equation (5) where the unit vector $\hat{\mathbf{n}}$ is directed outward and is normal to the surface element da , q are charges and ϵ_o is capacitivity. Accordingly,

$$\iint_a \mathrm{D} \cdot \hat{\mathrm{n}}\, da = q \tag{1.11}$$

where $\mathbf{D} = \epsilon_o \mathbf{E}$.

These two observations suggest that, the divergence of **D will** be non-zero. It is clear that equation (1.11) will become upon integrating and transforming, the following differential equation namely:-

$$\nabla \cdot \mathrm{D} = \rho_{\mathrm{e}} \tag{1.12}$$

where ρ_e is the charge density. This is evident that D is a flow vector which is analogous to magnetic vector **B**. Equation (1.12) makes sense because the electric charge density acts as a flow source for the vector **D**. If ρ_e does not vary with time, equation (1.12) is sufficient to define **D**. But if $\rho_e = 0$, then to define **D** we must satisfy in addition to the equation

$$\nabla \cdot \ \boldsymbol{J} + \frac{\partial \boldsymbol{\rho_e}}{\partial \boldsymbol{t}} = \mathbf{0} \tag{1.13}$$

and this is the continuity equation which has been satisfied.

Combining equation (1.12) with (1.13) one obtains

$$\nabla \cdot \ \boldsymbol{J} + \frac{\partial \mathbf{D}}{\partial \boldsymbol{t}} = \mathbf{0} \tag{1.14}$$

Equation (1.14) implies that, $\frac{\partial \mathbf{D}}{\partial \boldsymbol{t}}$ is some kind of current density and this is called the displacement current which is the mode of current transport in space, e.g. between capacitor plates. Thus equation (1.14) makes clear that any source for **J** is a sink of equal intensity for $\frac{\partial \mathbf{D}}{\partial t}$.

Comparing equation (1.9) and equation (1.14), it is evident that both the vector **B** and the total current density $\boldsymbol{J} + \frac{\partial \mathbf{D}}{\partial t}$ have no flow sources because their respective divergence is zero. However, this does preclude the existence of related vortex sources. A vector will have a vortex source if the curl is non-zero. We therefore define Magnetic field intensity vector **H** as the quantity which has a vortex source sum of equation (1.14). We now have

$$\nabla \times \boldsymbol{H} + \frac{\partial \boldsymbol{D}}{\partial t} = 0 \qquad (1.15)$$

In case of magnetostatics, there will be only steady currents in the absence of changing electric field and equation (1.15) becomes

$$\nabla \times \boldsymbol{H} = \mathbf{J} \qquad (1.16)$$

In summary, the $Maxwell^t s$ equations in integral form are -:

$$\oint E \cdot dl = -\int_s \frac{\partial B}{\partial t} da \qquad (1.17)$$

$$\mathbf{H} \cdot \mathbf{dl} = \int_s (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} da) \qquad (1.18)$$

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$$\int_s \mathrm{D} \cdot d\mathrm{l} = \int_v \rho dv \qquad (1.19)$$

$$\mathbf{B}.da = 0 \qquad (1.20)$$

Equations (1.17) to (1.20) can be transformed into differential equivalent by applying divergence and $Stoke^t s$ theorems to obtain time independent $Maxwell^t s$ equations in differential form as:-

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (1.21)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad (1.22)$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.23}$$

$$\nabla \cdot \mathbf{D} = 0 \tag{1.24}$$

where equation (1.24) is obtained from the fact that, $\nabla \cdot \mathbf{D} = \rho_e$ but if $\rho_e = 0$ then $\nabla \cdot \mathbf{D} = 0$.

1.3 Steady incompressible flow

If various points of the flow field all quantities such as velocity, density and pressure remain unchanged with time and the motion is said to be *steady* otherwise it is called *unsteady*. Accordingly, various quantities of the flow field become functions of space coordinates only because time drops out of the independent variables. If the density is constant throughout the flow field, it is said to be *incompressible*. Such an assumption is valid for liquid and also for gases at low speeds $M << 1$, where M is Mach number. Generally, the viscosity of a fluid depends on the temperature. It is known that for most of the incompressible fluids the viscosity can be treated as a constant. This assumption is of paramount importance because of such situations the velocity field does not depend on the temperature field. Hence, the equation of continuity and equation of motion can be first solved for the three velocity components and the pressure p and then, the results so obtained can be used to solve the equation of energy to determine the desired temperature field.

1.4 Background material on the problem

The phenomenal growth of electromagnetic brakes, MHD power generators, flow meter and electromagnetic pump in the recent years has been attracting considerable attention all over the world. It is obvious that in order to utilize MHD energy to a maximum, one should have a complete and precise knowledge of the amount of perturbations needed to generate MHD currents.

For example, there are two technological applications of MHD, which may both become very important in future. First, strong magnetic fields may be used to confine rings or columns of hot plasma that will be held in place long enough for thermonuclear fusion to occur and for net power to be generated. In the second application, which is directed towards a similar goal, liquid metals are driven through a magnetic field in order to generate electricity. In electromagnetic brake, the magnetic fields lines are partially dragged by the fluid, bending them (as embodied in tension force $\nabla\times\mathbf{B}=\mu_0\mathbf{J}$) so they can exert a decelerating tension force $\mathbf{J}\times\mathbf{B}=(\nabla\times\mathbf{B})\times\mathbf{B}/\mu_0=\mathbf{B}\cdot\nabla\mathbf{B}/\mu_0$ on the flow, where μ_0 is the vacuum permeability. This is the electromagnetic brake. The pressure gradient, which is trying to accelerate the fluid, is balanced by the magnetic tension. The rate of work done (per unit volume) by the pressure gradient, $v\cdot(-\nabla p)$, is converted into heat through viscous and Ohmic volume.

The MHD power generator, is similar to electromagnetic brake except that an external load is added to the circuit. Useful power can be extracted from the flow. This may ultimately be practical in power stations where a flowing conducting fluid can generate electricity directly without having to drive a turbine.

Finally for electromagnetic pump, can attach a battery to the electrodes and allow the current to flow. This produces Lorentz forces which either accelerates or decelerate the flow depending on the direction of the magnetic field. This method is used to pump liquid sodium coolant around a nuclear reactor. In this thesis, the case of the electromagnetic pump where there is a constant pressure gradient and transverse magnetic field is considered.

There has been an extensive literature on porous media, parallel infinite plates, transverse magnetic field, liquid through semi-infinite plate and infinite plates. Little has been done by considering all the fluid property variables. Attia, (2005) considered unsteady flow of a dusty conducting fluid between parallel plates with temperature dependent viscosity, while Attia, (2007) studied the effectiveness of the variable physical properties on the transient hydromagnetic Couette-Poiseuille MHD flow. Attia,(2006a) studied time varying hydromagnetic Couette flow with heat transfer of dusty fluid in the presence of uniform suction and injection considering the Hall effect and all these were solved numerically by finite difference method. Singh, (1993) studied hydromagnetic steady flow of viscous incompressible fluid between parallel infinite plates under the influence of transverse magnetic field. The resultant differential equation obtained was solved by method of solution of linear differential equation with constant coefficient. The results obtained can be applied to the designs and operations of MHD generators, MHD pump, electromagnetic flow meter and crude oil purification. In this work, steady and unsteady laminar viscous incompressible MHD fluid flow under the influence of transverse magnetic field is studied. Both Poiseuille flow and Couette fluid flo and the resultant partial differential equations are solved analytically or numerically. The resulting governing partial differential equations are solved analytically by Sumudu Transform for the linear differential equation with constant coefficients. The motion of two dimensional steady Poiseuille laminar flow of a viscous Magnetohydrodynamic incompressible fluid between two infinite parallel porous plates under the influence of uniform transverse magnetic field and with constant pressure gradient is also studied. Both the lower plate and the upper plates are assumed porous where the fluid entered the flow region through the lower plate and left through the upper plate with constant velocity. The resulting coupled differential equations were solved numerically by using finite difference approach. Finally, unsteady MHD Couette laminar flow of viscous incompressible fluid between two in- finite parallel porous plates in presence of uniform magnetic field is analyzed. The upper and lower plates are maintained at two different but constant temperatures T_2 and T_1 respectively, with $T_2 > T_1$. The upper plate is considered to be moving with constant velocity U_o while the lower plate is kept stationary. A sudden uniform and a constant pressure gradient and external uniform magnetic field with magnetic flux density vector $\mathbf{B}_o$ is applied in the positive y- direction which is assumed to be also the total magnetic field. The flow is subjected to a uniform suction from above and uniform injection from below at $t = 0$. The linear differential equations resulting from these are to be solved numerically by using finite difference approach under some given initial and boundary conditions so as to determine the velocity and temperature distributions for different values of the parameters M, S and α.

The Crank-Nicolson implicit method is used here at two successive time levels where the finite difference equations relating to the variables are obtained by writing the equations at the midpoint of the computational cell and subsequently replacing the different terms by their second order central difference approximation in the y - direction. On the other hand, the diffusion terms are replaced by the average of the central differences at two successive time-levels. The resulting block tri-diagonal system is solved using Thomas-algorithm.

1.5 Statement of the problem

The problem to be addressed in this research is that, when unsteady or steady laminar flow of incompressible viscous electrically conducting fluid between two infinite parallel nonconducting plates located apart under the influence of transverse magnetic field, the induced magnetic field appears to perturb the original magnetic field and also perturbs the original motion and induced electric field appears. These two are the basics of MHD. The plates extends from $-\infty < x < \infty$ and $-\infty < z < \infty$

This study intends to give an approximate solutions to the shape of velocity profiles and temperature distributions which is to be obtained either analytically by Sumudu transform and or numerically by finite difference approach for non porous and porous channels.

1.6 Objectives of the study

The major objectives of this study are to analyze the effect of magnetic field on viscous incompressible electrically conducting fluid between two infinite parallel porous plates for both Couette and Poiseuille fluid flow. Both steady and unsteady hydromagnetic incompressible fluid flow is considered. The present research will extend the work of (Singh, 2000).

The specific objectives of the study are-:

1. To investigate velocity distributions on the Couette MHD flow between

infinite parallel plates under the influence of transverse magnetic field by using Sumudu Transform.

2. To determine the effect of magnetic field on Poiseuille MHD fluid between infinite parallel porous plates with constant pressure gradient.
3. To determine the effect of porosity on unsteady MHD Couette flow when both plates are porous and are subjected to a uniform suction from above and uniform injection from below.
4. To obtain fluid velocity and temperature distribution for unsteady MHD Couette flow with heat transfer when both plates are maintained at different but constant temperatures.

1.7 Research Methodology

In this thesis, a theoretical framework for a quantitative analysis of Couette and Poiseuille MHD flow under the influence of transverse magnetic field of Newtonian fluids is provided. The present research will advance the quantitative understanding of the aspects of heat transfer associated with the movement of a surface under influence of transverse magnetic field. We will then formulate partial differential equations and proceed to solve them by either analytical or numerical methods and then find the accurate velocity field of the Couette MHD flow of Newtonian fluids. Analytically, Sumudu Transform will be used and numerically, finite difference method will be used to obtain velocity profiles and temperature distribution. The inclusion of the Newtonian fluids in the study will be intended to provide the data for exact solutions of the problem and the analytical method will be employed for all applicable situations. Later, the problem of heat transfer on unsteady incompressible flows of Couette MHD flow through parallel porous infinite plates under the influence of transverse magnetic field is solved. The energy equation will be solved numerically to obtain the temperature distribution. Finally, we will analyze the result, draw velocity profiles and temperature distributions and make conclusions.

1.8 Significance of the study

The areas of application of this research are broad and diverse because the study is fundamental. In considering the applicability of the research work, it is hoped that the results of the research is not only of fundamental interest, but also will advance the qualitative understanding of the aspects of heat transfer in MHD and industrial processes and also will provide appropriate design of parameters of the systems related to manufacturing process. This application where fluids flows under the influence of magnetic field may be applied in engineering problems like the manufacturing of MHD generators, MHD pumps, electromagnetic flow meter and to crude oil purification among others.

In MHD generators, they require very high temperatures to function than the convectional electric generators and thus they have high thermal efficiency for power plants. The MHD power generator is similar to electromagnetic brake except that an external load is added to the circuit. Useful power can be extracted from the flow. This may ultimately be practical in power stations where a flowing conducting fluid can generate electricity directly without having to drive a turbine.

Similarly, for electromagnetic pump, we can attach a battery to the electrodes and allow the current to flow. This produces Lorentz forces which either accelerates or decelerate the flow depending on the direction of the magnetic fluid. This method is used to pump liquid sodium coolant around a nuclear reactor (liquid-metal cooling). For example, in 1990’s Mitsubishi built a boat the *Yamoto* which uses a MHD drive, driven by helium, a cooled superconductor which can travel at 15 Km/h. In the present work, we will consider the case of the electromagnetic pump where there is a constant pressure gradient $P = dp/dx$ *with* transverse magnetic field.

In Meteorology, at times some of the meteorological problems involve differential equations that are difficult to be solved directly by applying boundary conditions. In such cases the processes of evaluating initial conditions and application of Sumudu transform as will be applied in this work will turn out to be an additional tool for solution of such problems. For example, the

Sumudu Transform could be applied to the linearized vorticity equation to derive the condition necessary for the barotropic instability. Moreover, it is evident that the behavior of the atmosphere can be analysed and understood in terms of the basic laws and concepts of physics. The fields of physics which are most applicable to the atmosphere are thermodynamics, radiation and hydrodynamics. In the atmosphere, the short wave radiation from the sun and the long wave from the earth travel as electromagnetic waves while hydrodynamics refer to the atmospheric motions and the associated forces. The results from the present study can be used to study the atmospheric MHD.

In Magnetic Drug targeting, this can find application in cancer patients as a precise method for delivery of medicine to the affected areas. The method involves the production of medicine to biologically compatible magnetic particles e.g. ferrofluids which are guided to the target through a careful placement of permanent magnets on the external body. MHD equations and finite element analysis may be used to study the interaction between the magnetic fluid particles in the bloodstream and external magnetic fields.

Fluid flow and heat transfer with moving boundary of liquid has an application in continuous casting, hot rolling, cooling of an infinite metallic plate in a cooling bath, polymer processing and the boundary layer along a liquid film in condensation process.

CHAPTER TWO

LITERATURE REVIEW

2.1. Literature Review

The flow field with heat transfer of viscous incompressible electrically conducting fluid between two parallel plates is a classical problem that has important applications in MHD power generators and pumps, accelerators, aerodynamics heating, electrostatic precipitation, polymer technology, petroleum industry. The problem has been considered by many researchers under different physical effects (Tani, 1962; Cramer and Pai, 1973 and Attia, 1998). Most of these studies are based on the constant physical properties, although some physical properties are varying with temperature and assuming constant properties is a good approximation as long as small differences in temperature are involved (Herwing & Wicken, 1986).

More accurate prediction for the flow and heat transfer is achieved by considering the variation of these physical properties with temperature. Attia, (2005), considered unsteady flow of a dusty conducting fluid between parallel plates with temperature dependent viscosity. He studied laminar flow of an electrically conducting viscous incompressible fluid and heat transfer between parallel non-conducting porous plates. The fluid was flowing between two electrically insulating infinite plates maintained at two constant but different temperatures. An external magnetic field was applied perpendicular to the plates. The magnetic Reynolds number was assumed small so that the induced magnetic field is neglected. The fluid was acted upon by constant pressure gradient and viscosity assumed to vary exponentially with temperature. The governing coupled non-linear partial differential equations were solved numerically using finite difference approximation. The system was solved using the Crank-Nicolson implicit method. The solution was also given in terms of Gr, Re, Pr, Ha and Ec numbers.

The problem of unsteady laminar fully-developed flow and heat transfer of an electrically conducting and heat generating or absorbing fluid with variable properties through porous channels in the presence of uniform magnetic and electric fields was formulated by Ali, (2001). The general governing equations which included such effects as magnetic field, electric field, porous medium inertia and heat generation or absorption effects were non-dimensionalized and solved numerically by the implicit finite-difference methodology. A representative set of numerical results for the transient and steady-state velocity and temperature profiles, the skin friction coefficients at both the upper and lower walls of the channel as well as the heat transfer coefficient at the lower wall were presented graphically and discussed. It was found that, both the magnetic field and the porous medium caused lower velocity distributions and skin-friction coefficients while the presence of the electric field produced higher velocity distributions and skin-friction coefficients. The presence of a heat generating source resulted in higher steady state temperatures and lower velocities due to variable properties. The lower wall heat transfer decreased due to heat generation for most of the transient stages while its steady-state value increased. On the other hand, a heat-absorption sink produced, in general, lower temperatures, and higher velocities and heat transfer at the lower wall. Increases in the variable viscosity exponent caused higher velocities and lower skin-friction coefficients.

The effectiveness of the variable physical properties on the transient hydromagnetic Couette-Poiseuille flow was studied by Attia, (2007). He considered transient hydromagnetic Couette-Poiseuille flow and heat transfer on an electrically conducting fluid in presence of a transverse magnetic field with variable physical properties. The fluid was subjected to a constant pressure gradient and external magnetic fields perpendicular to the plates were kept at different but constant temperatures. The coupled nonlinear partial differential equations of motion and the energy equation including the viscous and joule dissipation terms under the initial and boundary conditions were solved numerically using finite difference approximation to obtain the velocity and temperature distribution at any instant time and expressed in terms of *Gr*, *Re*, *Pr*, *Ha* and E_C numbers. Time varying hydromagnetic Couette flow with heat transfer of dusty fluid in the presence of uniform suction and injection considering the Hall effect was extensively analysed by Attia, (2006a).

Different pressure gradients have been applied in the MHD under steady and laminar flows. For example, Drake, (1965) studied the flow in a channel due to periodic pressure gradient. Singh and Ram (1978) considered unsteady MHD flow in a channel under variable pressure gradient, while Singh,(2000) studied unsteady flow of liquid through a channel with pressure gradient changing exponentially under the influence of transverse magnetic field.

Attia, (2006b) studied ion effect on unsteady Hartmann flow with heat transfer under exponentially decaying pressure gradient. On the other hand, Singh, (1993) studied hydromagnetic steady flow of viscous incompressible fluid between parallel infinite plates under the influence of inclined magnetic field. The differential equation obtained was solved by method of solution of linear differential equation with constant coefficient. The results obtained were applied to the designs and operations of MHD generators, MHD pump, electromagnetic flow meter and crude oil purification.

Singh and Okwoyo, (2008) studied steady laminar flow of viscous incompressible fluid between two parallel infinite plates when the upper plate is moving with constant velocity and lower plate held stationary under the influence of transverse magnetic field. The resulting differential equation was solved by the application of Laplace transform and analytical expression obtained.

Umavathi et al., (2010) analyzed Poiseuille-Couette flow of two immiscible fluids between inclined parallel plates. One of the fluids was assumed to be electrically conducting while the other fluid and channel walls were assumed to be electrically insulating. This was investigated analytically by regular perturbation method and numerically by finite difference technique. The equations for velocity and temperature distribution were solved numerically and the results depicted graphically.

Das et al. (2012) studied unsteady free convective flow and heat transfer in a viscous incompressible electrically conducting fluid past a vertical porous plate through medium with time dependent permeability and the presence of a transverse magnetic field. The solution for velocity and temperature of the flow field were obtained. The results obtained were discussed interms of Grashof number $Gr > 0$ corresponding to cooling the plate, and effect of Prandtl number, Pr on the temperature distribution of the flow studied.

Manyonge et al., (2012) examined a motion of two dimensional Poiseuille steady flow of a viscous MHD incompressible fluid flowing between two infinite parallel porous plates under the influence of transverse magnetic field with constant pressure gradient. The effect of velocity if the lower plate is porous was also accessed. The resulting differential equation was solved by an analytical method and the solution expressed in terms of Hartmann number.

From the literature review, much has not been done on Couette and Poiseuille MHD flows resultant partial differential equations being solved by trasform methods other than Laplace Transform. Hence, the need to use Sumudu Transform to approximate solutions which can be used to solve such physical problems on porous and nonporous channels under the influence of transverse magnetic fields.

CHAPTER THREE

MATHEMATICAL FORMULATIONS OF THE PROBLEM

3.1 Description of the governing equations

The basic fluid equations needed for the problem under investigation are the conservation of mass or Continuity equation and the conservation of momentum also called Navier-Stokes equations. The Continuity equation requires that the mass of fluid entering a fixed control volume either leaves that volume or accumulates within it. It is thus a "mass balance" requirement posed in mathematical form, and is a scalar equation. The momentum equation may be thought of as a "momentum balance". These are vector equations, i.e. there is a separate equation for each of the coordinate directions and they are the fluid dynamics equivalent of Newton's second law which is given as $F = ma$, where F *is* force, m mass while a is acceleration. In situations where the fluid may be treated as incompressible and temperature differences are small, the continuity and momentum equations are sufficient to specify the velocities and pressure, i.e. four equation and four unknown quantities. If the flow is compressible (density is not constant), or if heat flux occurs (temperature not constant), at least one additional equation and often, the energy equation is used. These equations may be used to analyze the flow of most common fluids in internal (e.g. pipes) or external flow situations.

3.2 Mathematical expression for Navier-Stokes Equation

Shear stresses are present because of fluid viscosity and are caused by the transfer of molecular momentum. The friction force τ *is* assumed to be proportional to the coefficient of viscosity μ and the rate of angular deformation. The Navier-Stokes equations (sometimes abbreviated as N-S equations) are the equations which govern the three dimensional incompressible flow and are expressed as follows:-

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{q} \qquad (3.1)$$

where $\nu = \dfrac{\mu}{\rho}$ is the kinematic viscosity.

Equation (3.1) in the Navier –Stokes equation in vectorial form can be written as

$$\frac{Du}{Dt} = F_x - \frac{1}{\rho}\nabla p + \nu\nabla^2 u \qquad (3.2)$$

$$\frac{Dv}{Dt} = F_y - \frac{1}{\rho}\nabla p + \nu\nabla^2 v \qquad (3.3)$$

$$\frac{Dw}{Dt} = F_z - \frac{1}{\rho}\nabla p + \nu\nabla^2 w \qquad (3.4)$$

where $\tilde{q}$ is the velocity of the fluid particle, $\tilde{F}$ is the external force per unit mass acting on the fluid, ρ is the density of the fluid, p is the pressure at a point on surface element ds which has an outward unit normal $\hat{n}$ and F_x, F_y and F_z are the components of force in x,y and z directions respectively.

The Navier-Stokes equations are thus second order non-linear partial differential equations. Until the present day there exists no general method for solving these equations. This is particularly true when friction and inertial forces are of the same order of magnitude in the entire flow, so that neither can be neglected. Analytical (exact) solutions have therefore been attempted only for flows with relatively simple geometry. Even such solutions are based on idealizations such as infinite plates, infinitely long cylinders, fully developed parallel flow in pipe, certain types of unsteady flows such as a plate vibrating in a fluid etc. In these cases, the equations are made linear by taking a simple geometry of flow and assuming the fluid to be incompressible. Hence, these exact solutions hold well in a particular region of a real problem. As restricted as they are, these exact solutions are very useful and add greatly to our knowledge of the flow of real or Newtonian fluids.

In this study, buoyancy effects are not considered since the channel is not vertically oriented and acted upon by gravity, hence, the effect of body forces are neglected, and the terms with gravity (which is the most common source of the body forces) will vanish in the above equations. In case of MHD fluid flows, the effect of magnetic forces in the fluid flows as the only body forces present is considered.

3.3 Dimensionless Parameters

Fluid mechanics equations are typically cast in dimensionless form so that the relative strengths of the different terms can be inferred by the size of any multiplying factors. The equation of motion

$$\frac{\partial \mathrm{u}}{\partial t}+(\mathrm{u}\cdot\nabla)\mathrm{u}=-\frac{\nabla \mathrm{p}}{\rho}+\nu\,\nabla^2\mathrm{u}+\frac{\mathrm{J}\times\mathrm{B}}{\rho}$$

can be written in dimensionless form by making the substitutions:

$$J^*=\frac{J}{\sigma u_o \mathrm{B_o}} \tag{3.5}$$

$$p^*\frac{\mathrm{p}}{u_o\sigma u_o \mathrm{B_o^2}a} \tag{3.6}$$

$$\nabla^*=\frac{1}{a}\nabla,\quad U^*\frac{\mathrm{u}}{u_o},\quad \mathrm{B}^*=\frac{\mathrm{B}}{\mathrm{B_o}} \tag{3.7}$$

where a, u_O and $\mathbf{B}_O$ are characteristic values of length, velocity and applied magnetic fields respectively. Using this system, the equation of motion (excluding external forces) becomes:

$$\frac{1}{N}\left(\frac{\partial u^*}{\partial t}+(u^*\cdot\nabla)u^*\right)=-\nabla p^*+J^*\times B^*+\frac{1}{Ha^2}\nabla^2 u^* \tag{3.8}$$

The characteristic parameters Ha and N *are* the Hartmann number and the interaction parameter respectively. They are defined as -:

$$Ha=a\ \mathrm{B_o}\sqrt{\frac{\sigma_\mathrm{f}}{\mu_\mathrm{f}}} \tag{3.9}$$

$$\mathrm{N}=\frac{\mathrm{Ha}}{\mathrm{Re}} \tag{3.10}$$

$$\mathrm{Re}=\frac{\rho u_o a}{\mu_\mathrm{f}} \tag{3.11}$$

where Re is Reynolds number.

When the Hartmann number and the interaction parameter are both sufficiently large, the momentum equation (3.8) throughout the bulk of the fluid can be reduced to the simple form

$$\nabla p = \mathrm{J} \times \mathrm{B} \tag{3.12}$$

(a) Reynolds number

This is the ratio of inertial forces to viscous forces. It is a parameter for viscosity. If l is the characteristic length, t time, then mass of an element is proportional to ρl^3 and acceleration is $\frac{1}{t^2}$. Then the inertial force,

$$\mathbf{F}_{\mathrm{i}} \propto mass \times acceleration$$

$$\mathbf{F}_{\mathrm{i}} \propto \rho l^3 \times \frac{l}{t^2} \tag{3.13}$$

$$\mathbf{F}_{\mathrm{i}} \propto \rho l^2 \times \frac{l^2}{t^2} \tag{3.14}$$

and since $V = \frac{1}{t}$,

$$F_i \propto p l^2 V^2 \tag{3.15}$$

On the other hand viscous force

$$\mathbf{F}_{\mathrm{v}} \propto viscous\ shear\ stress \times area \tag{3.16}$$

$$\mathbf{F}_{\mathrm{v}} \propto \mu \times velocity\ gradient \times l^2 \tag{3.17}$$

and since velocity gradient,

$$\mathbf{V}_{\mathrm{g}} \propto \frac{v}{l} \tag{3.18}$$

Then viscous force

$$\mathbf{F}_{\mathrm{v}} \propto \mu \times \frac{V}{l} l^2 \tag{3.19}$$

$$\mathbf{F}_{\mathrm{v}} \propto \mu V l \qquad (3,20)$$

Thus,

$$\frac{inetia\ \ force}{viscous\ \ force} \propto \frac{\rho V^2 L^2}{\mu V L} \qquad (3.210$$

$$\mathrm{Re} = \frac{\rho V L}{\mu} \qquad (3,22)$$

or

$$\mathrm{Re} = \frac{VL}{\nu} \qquad (3,23)$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, V_g is the velocity gradient, F_i is the inertia force, F_v is the viscous force and Re is the Reynolds number.

If the Reynolds number is small, the viscous forces will be predominant and the effect of viscosity will be felt in the whole flow field and hence we can ignore the inertia force. If *Re* is large, the inertial forces will be predominant then we can neglect the effect of viscous force and in such a case the effect of viscosity can be considered to be confined within the layer, known as boundary layer, adjacent to a solid boundary and consequently the fluid may be treated as non-viscous fluid. However, if *Re* is very large the flow ceases to be laminar and becomes turbulent. The Reynolds number at which the transition from laminar to turbulent occurs is known as critical *Reynolds number*. In laminar flow, the individual streamlines run in an orderly manner side by side, while in turbulent flow, the streamlines are interwoven with each other in an irregular manner.

(b) Hartmann number

Using L_O and V_O as a scale length, velocity and introducing a new scale field $\mathbf{B}_O$, comparable with the actual magnetic field in a real engineering system, it can be inferred that

(i) Magnetic viscous force per unit volume

$$\sim \delta(\sigma B_o^2 V_o) \qquad (3.24)$$

(ii) Hydrodynamic viscous force per unit volume

$$\sim \delta\rho \frac{V_o}{L_o^2} \qquad (3.25)$$

The ratio of magnetic viscous and hydrodynamic viscous forces leads to the Hartman number, *Ha*, viz

$$Ha^2 = \frac{Magnetic\ viscous\ forces}{hydrodynamic\ viscous\ forces}$$

or

$$Ha^2 = L_o^2\ B_o^2 \frac{\sigma}{\mu_f} \tag{3.26}$$

(c) Prandtl number

This expresses the ratio of momentum diffusivity (kinetic viscosity) to thermal diffusivity. It is named after German phycist Ludwing Prandtl. It is defined as

$$\mathrm{Pr} = \frac{\nu}{\alpha} \tag{3.27}$$

$$Pr = \frac{viscous\ diffusion\ rate}{thermal\ diffusion\ rate} \tag{3.28}$$

or

$$\mathrm{Pr} = \frac{C_p \mu}{k} \tag{3.29}$$

where

ν = kinematic viscosity,

α = thermal diffusivity,

k = thermal conductivity,

C_p = specific heat at constant pressure.

In heat transfer problems the Prandtl number controls the relative thickness of the momentum and thermal boundary layers. When Pr is small, it means that the heat diffuses very quickly compared to the velocity (momentum). This means that for a liquid metal, the thickness of the thermal boundary layer is much bigger than the velocity boundary layer

(d) Nusselt Number

This is the ratio of convective to conductive heat transfer across the boundary (surface). It is derived from the Newton's law of cooling (convective terms) and heat conduction terms (at the same condition as the heat convection). Nusselt number Nu is therefore defined as

$$Nu = \frac{Convective\ heat\ transfer}{Conductive\ heat\ transfer} \tag{3.30}$$

$$Nu = \frac{h(T_w - T_\infty)}{\lambda \frac{(T_w - T_\infty)}{L}} \tag{3.31}$$

$$Nu = \frac{hl}{\lambda} \tag{3.32}$$

Where h = Convective heat transfer coefficient (w/m^2k)

λ = Thermal conductivity of the fluid ($w/m\ k$)

l = Characteristic length (m)

The Characteristic length l is determined by the direction of the growth (thickness) of the boundary layer.

If Nusselt number is $Nu = 1$, then the convection and conduction term have relatively similar magnitude and thus characterized by laminar flow. If Nu is large, this implies that the convective term is dominant which is typically characterized by turbulent flows (usually Nu number of range 100-1000). Thus by understanding Nu number, we can infer the dominance between convection heat transfer terms and conduction heat transfer terms thus enabling us to design better and more efficient thermal engineering, especially in the convective heat transfer field.

3.4 Steady incompressible flow of fluid with constant properties

3.4.1Flow between parallel plates (velocity distribution)

A very simple solution of the equations of motion (3.2) to (3.4) can be obtained for the flow between two parallel plates which are kept at a finite distance apart. Equation of continuity is given as:-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.33}$$

The assumption made here is that, x - axis is along the direction of the flow, y - axis being at right angle to it, and the width of the plates, parallel to z - direction be large compared to the distance between the plates. The motion is in the two dimensional and therefore, all the variables will be independent of z - coordinates. Hence,

$$\frac{\partial}{\partial z}(.) \equiv 0, \quad u = u(y), \quad v = 0, w = 0 \text{ and } p = p(x) \tag{3.34}$$

Steady flow of a viscous incompressible fluid in absence of body forces between two infinite parallel plates situated at $y = 0$ and $y = h$ and the flow is along the x- axis is taken parallel to the plates is considered. Figure 3.1 below shows velocity distribution between two parallel plates.

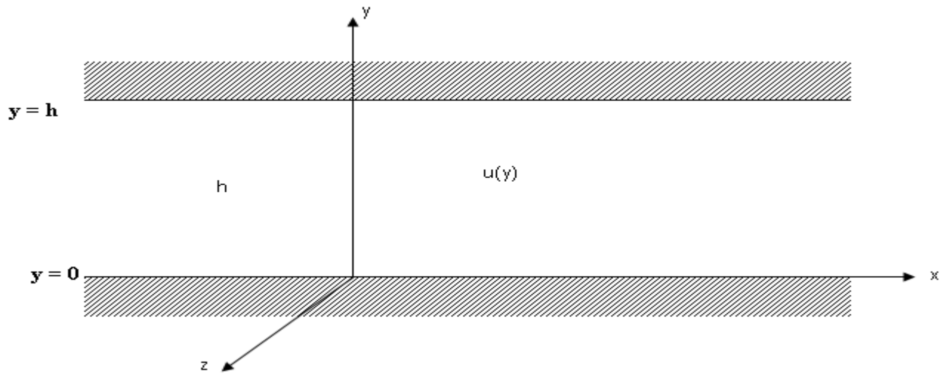

Figure 3.1. Velocity Distribution between parallel plates

For an incompressible fluid, equation of conservation of mass (continuity) is given as

$$\nabla . q = 0 \tag{3.35}$$

For two dimensional flows, equation (3.35) will be

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.36}$$

Since the plates are of infinite length, therefore all the terms will be independent of x except pressure.

Thus

$$\frac{\partial u}{\partial x} = 0 \tag{3.37}$$

(u is not a function of x it is independent of x) and

$$\frac{\partial v}{\partial y} = 0 \tag{3.38}$$

(v is not a function of y and $v = 0$) since there is no flow motion along y - direction.

The Navier-Stokes equation in absence of body forces in now given as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{3.39}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{3.40}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{3.41}$$

For two dimensional motion, $w = 0$, and $z = 0$, therefore equations of motion are reduced to the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.42}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3.43}$$

For steady motion

$$\frac{\partial u}{\partial t} = 0 \quad \text{and} \quad \frac{\partial v}{\partial t} = 0 \tag{3.44}$$

Since the plates are of infinite lengths, then from equation (3.36), (3.37) and (3.38) the momentum equation (3.42) and (3.43) respectively becomes

$$0 = -\frac{\partial p}{\rho\ \partial x} + v\frac{\partial^2 u}{\partial y^2} \tag{3.45}$$

$$0 = -\frac{\partial p}{\rho\ \partial y} \tag{3.46}$$

Thus the continuity equation $\frac{\partial u}{\partial x} = 0$ is used in obtaining the equation (3.45) and (3.46) and conclude that u is a function of y only and p is a function of x only. From equation (3.46) we find that, p does not depend on y. Using equation

(3.45) and differentiating it with respect to x one gets,

$$\frac{d^2 p}{dx^2} = 0 \tag{3.47}$$

where total differentiation has been taken as p does not depend on y

On integrating equation (3.47) gives

$$\frac{dp}{dx} = -\mathrm{P} \tag{3.48}$$

where P is a constant.

Thus (3.45) can be written as

$$0 = \frac{-1}{\rho}(-P) + v\frac{d^2u}{dy^2} \tag{3.49}$$

or

$$\frac{d^2u}{dy^2} = \frac{1}{\rho v}P \tag{3.50}$$

where $v = \frac{\mu}{\rho}$, so equation (3.45) may be written as

$$\frac{d^2 p}{dy^2} = \frac{-1}{\mu}\mathrm{P} \tag{3.51}$$

where

$$P = -\frac{dp}{dx} \tag{3.52}$$

from equation (3.48). On integrating (3.51) one gets

$$\frac{du}{dy} = \frac{-1}{\mu}Py + A \tag{3.53}$$

where A is a constant.

Again integrating (3.53) one obtains

$$u = \frac{-1}{2\mu}Py^2 + Ay + B \tag{3.54}$$

or

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + Ay + B \tag{3.55}$$

where A and B are constants of integration to be determined from boundary conditions of either plane Couette flow, plane Poiseuille flow or generalized plane Couette flow. Thus the shearing stress τ can we written using equation (3.55) as

$$\tau = \mu \frac{du}{dy} = \mu A - Py \tag{3.56}$$

3.4.2 Plane Couette flow

Couette flow is named in honour of Maurice Marie Alfred Couette, a Professor of Physics at the French University of Angers in the late 19^{th} century. Figure 3.2 depicts velocity distribution in a Plane Couette flow.

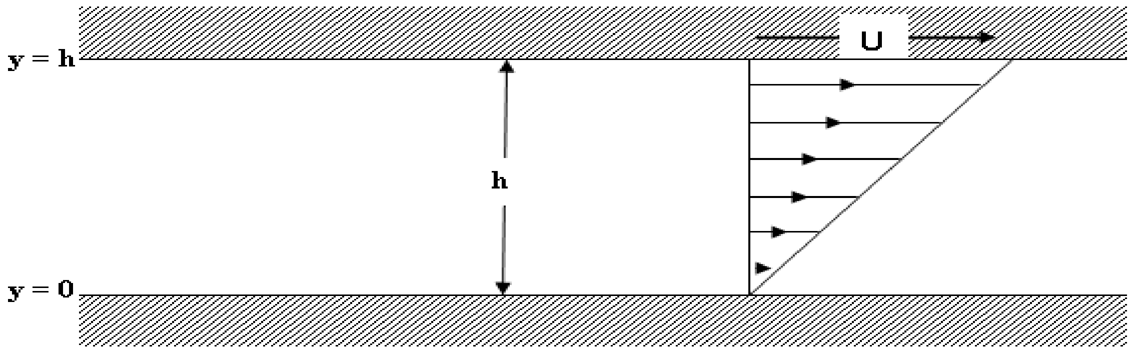

Figure 3.2 Velocity distribution in Plane Couette flow

This is also called shear flow in which the flow is between two parallel plates one which is at rest and the other moving with uniform velocity U in its plane and the pressure gradient is taken to be zero. Consider a laminar flow of viscous incompressible fluid between two infinite parallel plates separated by a distance h. If we let x be the direction of the fluid flow, y the direction perpendicular to the flow, and the width of the plates parallel to the z-direction. The word infinite here means that, the width of the plates is large compared with h and hence the flow may be treated as a two dimensional i.e. $\frac{\partial(\cdot)}{\partial x} = 0$. Let also the plates be long enough in the x-direction for the flow to be parallel. Here the flow is said to be parallel if only one velocity component is non-zero, all fluid particles moving in one direction. Hence, the velocity components v and w will be zero everywhere. Since the flow is steady the flow variables are independent of time i.e. $\frac{\partial(\cdot)}{\partial t} = 0$. Furthermore, the equation of continuity (3.35) reduces to (3.37) and (3.38) and conclude that $u=u(y)$. For the present problem,

$$u = u(y), \quad v = 0, \quad w = 0, \quad \frac{\partial(.)}{\partial z} = 0, \quad \frac{\partial(.)}{\partial t} = 0 \qquad (3.57)$$

For the two dimensional flow in absence of body forces, the Navier-Stokes equations for x and y directions (3.39) and (3.40) keeping equation (3.57) in mind are:

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \qquad (3.58)$$

$$0 = -\frac{\partial p}{\partial y} \qquad (3.59)$$

Equation (3.59) here shows that the pressure does not depend on y. Hence, p is a function of x alone and so equation (3.58) reduces to

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu} \times \frac{dp}{dx} \qquad (3.60)$$

Differentiating both sides of equation (3.60) with respect to x gives

$$0 = \frac{1}{\mu} \frac{d^2 p}{dx^2} \qquad (3.61)$$

or

$$\frac{d}{dx}\left(\frac{dp}{dx}\right) = 0 \qquad (3.62)$$

so that

$$\frac{dp}{dx} = \text{constant} = P(\text{say}) \qquad (3.63)$$

Hence, equation (3.60) reduces to

$$\frac{d^2 u}{dy^2} = \frac{P}{\mu} \qquad (3.64)$$

Integrating equation (3.64) gives

$$\frac{du}{dy} = \frac{Py}{\mu} + A \qquad (3.65)$$

and integrating equation (3.65) again gives

$$u = Ay + B + \frac{Py^2}{2\mu} \tag{3.66}$$

where A and B are constants to be determined as earlier mentioned by the boundary conditions of the problem under consideration. Similarly, the plate $y = 0$ is kept at rest and the plate $y = h$ is allowed to move with velocity U. Hence the no slip condition gives rise to the boundary conditions:

$u = 0$ at $y = o$; and $u = U$ at $y = h$ (3.67)

Using the boundary conditions in (3.67) and then in equation (3.66) gives $B = 0$ and $U = \frac{1}{2\mu}Ph^2 + Ah \Longrightarrow Ah = U - \frac{1}{2\mu}Ph^2$ so that finally gives

$$A = \frac{U}{h} - \frac{1}{2\mu}Ph \tag{3.68}$$

and equation (3.66) reduces to

$$u = \frac{1}{2\mu}Py^2 + (\frac{U}{h} - \frac{1}{2\mu}Ph) \tag{3.69}$$

or

$$u = \frac{y}{h}U + \frac{1}{2\mu}Py\left[y - h\right] \tag{3.70}$$

Since the problem is plane Couette flow, then this means that, $P = 0$ and hence equation (3.70) becomes

$$u = \frac{y}{h}U \quad \text{or} \quad \frac{u}{U} = \frac{y}{h} \tag{3.71}$$

Thus equation (3.66) in non-dimensional form is given by equation (3.71) and the velocity distribution is linear as shown by figure 3.2.

Finally, the skin friction (or drag per unit area) is determined by $\sigma_{yx} = \mu\frac{\partial u}{\partial y} = \mu\frac{U}{h}$ by using equation (3.71).

3.4.3 Plane Poiseuille flow

Using equation (3.66) for the plane Poiseuille flow, both plates are kept at rest and the fluid is kept in motion by a pressure gradient P is non-zero, Raisinghania, (2010). Consider Figure 3.3 showing velocity distribution in a plane Poiseuille flow.

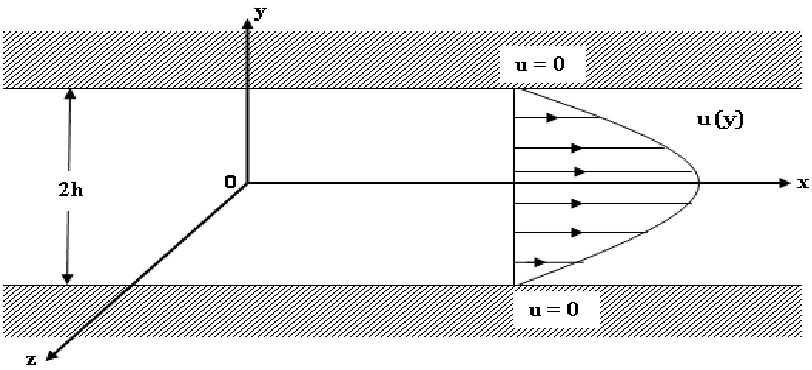

Figure 3.3 Velocity distribution in Plane Poiseuille flow

Let the distance between the plates be $2h$ and the axis for the sake of convenience be taken in the middle of the channel. In such a case, using the no-slip condition, the boundary conditions for the present problem are:

$$y = \pm h : \quad u = 0 \qquad (3.72)$$

Therefore equation (3.66) becomes

$$u = -\frac{1}{2\mu}\frac{dp}{dx}\left[y^2 - h^2\right] \qquad (3.73)$$

or

$$u = \frac{P}{2\mu}\left[y^2 - h^2\right] \qquad (3.74)$$

where $A=0$ and $B=\frac{-1}{2\mu}Ph^2$ and this shows that the velocity distribution is parabolicas shown in figure 3.3. If u_{max} denotes the absolute value of maximum velocity, then maximum velocity occurs in the middle of the channel which and is obtained by substituting $y = 0$. Thus

$$u_{\max} = \frac{h^2}{2\mu}\frac{dp}{dx} \quad \text{or} \quad u_{\max} = -\frac{h^2 P}{2\mu} \qquad (3.75)$$

and hence, the velocity distribution in a plane Poiseuille flow in non-dimensional form is given by

$$\frac{u}{u_{\max}} = 1 - \frac{y^2}{h^2} \qquad (3.76)$$

On the other hand, the average velocity distribution for the present flow and using (3.74) is given by

$$u_a = \frac{1}{2h}\int_{-h}^{h} u\, dy = -\frac{1}{2h}\times\frac{P}{2\mu}\int_{-h}^{h}\left[y^2 - h^2\right]dy \tag{3.77}$$

which on simplification it gives

$$u_a = -\frac{Ph^2}{3\mu} \tag{3.78}$$

and using (3.78) we have

$$P = -3\mu\times\frac{u_a}{h^2} \tag{3.79}$$

To find the shearing stress distribution in the flow σ_{yx} we use (3.74) to obtain

$$\sigma_{yx} = -\mu\frac{du}{dy} = -Py \tag{3.80}$$

and the skin friction at $y = h$ and using (3.79) we obtain

$$(\sigma_{yx})_{y=h} = \frac{3\mu\, u_a}{h} \tag{3.81}$$

Hence using (3.81), the frictional coefficient for laminar flow between two stationary
plates is given as:-

$$C_f = \frac{(\sigma_{yx})_{y=h}}{\frac{1}{2}\times\rho u_a^2} = \frac{3\mu\, u_a}{h}\times\frac{2}{\rho u_a^2} = 6\times\frac{\mu}{\rho h u_a} = \frac{6}{\mathrm{Re}} \tag{3.82}$$

where $Re = \frac{\rho h u_a}{\mu} = \frac{h u_a}{\nu}$

3.4.4 Generalized plane Couette flow

For the generalized plane Couette flow, the plate $y = 0$ is kept at rest and the plate $y = h$ is allowed to move with velocity U. Figure 3.4 shows velocity distribution in generalized Plane couette flow which like plane Couette flow but with non-zero Pressure gradient.

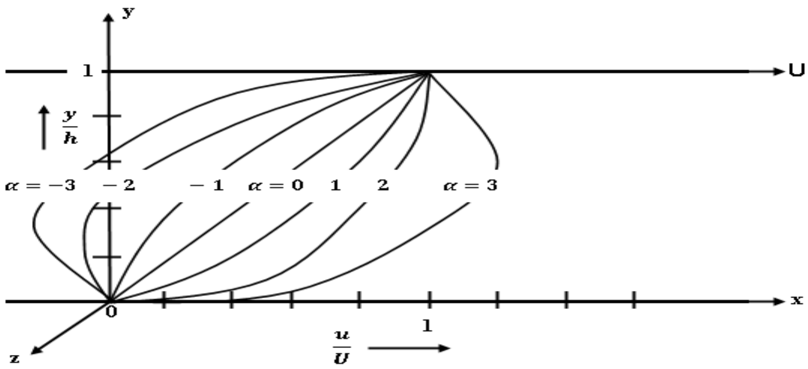

Figure 3.4 Velocity distribution in generalized Plane Couette flow

With no-slip condition this gives rise to boundary conditions which are the same as in plane Couette flow given in equation (3.67).

Using the boundary conditions given in equation (3.67) into equation (3.66) gives

$$0 = B, \quad U = Ah + B + \frac{Ph^2}{2\mu} \quad and \quad A = \frac{U}{h} - \frac{Ph}{2\mu} \tag{3.83}$$

Using equation (3.83) into equation (3.66), it gives

$$u = \frac{y}{h}U - \frac{Phy}{2\mu} + \frac{Phy^2}{2\mu} \tag{3.84}$$

or

$$\frac{u}{U} = \frac{y}{h} + \alpha\frac{y}{h}(1 - \frac{y}{h}) \tag{3.85}$$

where

$$\alpha = -\frac{h^2P}{2\mu U} \tag{3.86}$$

is the dimensionless pressure gradient.

Clearly, the velocity fields depend on the nature of the non-dimensionless pressure gradient and therefore different cases are possible:-

Case i $\alpha > 0$

If $\alpha > 0$, the pressure is decreasing in the direction of the motion and thus for favorable pressure gradient, the velocity distribution is positive over the whole width between the plates.

Case ii $\alpha < 0$

If $\alpha < 0$, the pressure is increasing in the direction of the flow. A back flow begins to occur near the stationary plate as $\alpha < -1$ which is due to the influence of the adverse pressure gradient which surpasses the dragging action of the fast layer on fluid particles in that region.

Case iii $\alpha = 0$

In this case zero pressure gradient is present and pressure is constant throughout the field flow to equation (3.66) reduces to equation (3.71) and in this case the velocity distribution is linear.

The velocity distribution as a function of the distance from the stationary wall for various values of α as shown in figure 3.4. It also indicates the arrangement of plates with coordinates axes $\frac{y}{U}$ and $\frac{y}{h}$ plotted along x-axis and y-axis respectively.

To determine the average and maximum velocities we proceed as follows:-

The average velocity distribution, u_a for generalized plane Couette flow is given by

$$u_a = \frac{1}{h}\int_0^h u\,dy = \frac{1}{h}\int_0^h \left[\frac{y}{h}U + \alpha U\left(\frac{y}{h} - \frac{y^2}{h^2}\right) \right] dy \tag{3.87}$$

$$u_a = \left(\frac{1}{2} + \frac{\alpha}{6}\right)U \tag{3.88}$$

and on simplification it gives

$$u_a = \frac{(\alpha + 3)}{6})U \tag{3.89}$$

If Q is the volumetric flow per unit time per width of the channel, then

$$Qhu_a = \frac{h(\alpha + 3)}{6})U \tag{3.90}$$

From equation (3.85)

$$\frac{du}{dy} = \frac{U}{h} + \frac{\alpha U}{h}\left(1 - \frac{2y}{h}\right) \tag{3.91}$$

Hence, for maximum or minimum velocity, we know that $\frac{du}{dy} = 0$, hence, equation (3.91) becomes

$$\frac{U}{h} + \frac{\alpha U}{h}\left(1 - \frac{2y}{h}\right) = 0 \tag{3.92}$$

which reduces to

$$\frac{y}{h} = \frac{1}{2}(1 + \frac{1}{\alpha}) \tag{3.93}$$

Hence, from (3.93) it follows that, the maximum velocity for $\alpha = 1$ will occur at $\frac{y}{h} = 1$, i.e. $y=h$ and the minimum velocity will occur for $\alpha = -1$at $\frac{y}{h} = 0$ i.e. $y=0$. This shows that, for $\alpha = -1$, the velocity gradient at the stationary will be zero and it becomes negative for some value of $\alpha < -1$. Hence the reverse flow will take place at $\alpha = -1$. Equation (3.93) breaks down when $-1 < \alpha < 1$ because the maximum and minimum values of $\frac{y}{h}$ have already been reached at $\alpha = 1$and $\alpha = -1$ respectively. Using equation (3.93) and equation (3.91) the maximum and minimum velocities are given by

$$U_{\max} = U\frac{(1+\alpha)^2}{4\alpha} \quad when \geq 1 \tag{3.94}$$

and

$$U_{\min} = U\frac{(1-\alpha)^2}{4\alpha} \quad when \leq 1 \tag{3.95}$$

To determine the shearing stress, skin friction and the coefficient of friction, we use equation (3.91), where the shearing stress distribution in the flow is given by

$$\sigma_{yx} = \mu\frac{du}{dy} = \frac{\mu U}{h}(1 + \alpha(1 - \frac{2y}{h})) \tag{3.96}$$

Using equation (3.60) and equation (3.67), the skin friction at the plates $y = 0$ and
$y = h$ are given by

$$\left[\sigma_{yx}\right]_{y=0} = \frac{\mu U}{h}(1+\alpha) = \frac{6\mu(1+\alpha)}{(3+\alpha)}u_a \tag{3.97}$$

$$\left[\sigma_{yx}\right]_{y=h} = \frac{\mu U}{h}(1-\alpha) = \frac{6\mu(1-\alpha)}{(3+\alpha)}u_a \tag{3.98}$$

Using equation (3.96), the coefficient of friction or the drag coefficient corresponding to $\left(\sigma_{yx}\right)_{y=0}$ is given by

$$C_f = \frac{\left[\sigma_{yx}\right]_{y=0}}{\rho u_a^2} = \frac{12\mu(1+\alpha)}{\rho h(\alpha+3)u_a} \tag{3.99}$$

If Reynolds number $= \mathrm{Re} = \dfrac{hv_a}{\nu}$. then

$$C_f = \frac{12(1+\alpha)}{\mathrm{Re}(\alpha+3)} \tag{3.100}$$

Similarly, the coefficient corresponding to $\left(\sigma_{yx}\right)_{y=h}$ is given by

$$C_f' = \frac{12(1-\alpha)}{\mathrm{Re}(\alpha+3)} \tag{3.101}$$

For practical purposes, the mean C_f and $C_f^{'}$ is employed to estimate the energy losses in parallel plates.

3.5. Temperature distribution in steady laminar incompressible flow with constant fluid properties

One of the main differences between the compressible and incompressible fluid flow for temperature distribution in steady, laminar flows is that, in compressible flow the equation of motion and energy equation are coupled whereas in an incompressible flow, with constant fluid properties *ρ*, *μ* and *k* the equation of motion and energy equation are uncoupled. Hence, in incompressible fluids, one can easily find temperature distributions since viscosity of fluids depends on temperature.

When dealing with incompressible fluid flow, the fluid properties such as density *ρ*, coefficient of viscosity *μ* and the coefficient of thermal conductivity *k* are nearly constant so that, the unknown quantities reduce to five (*u, v, w, p, T*) which are obtained with the help of Navier-Stokes equation and the energy equation. When dealing with temperature distribution for incompressible fluid flow, the equation of continuity and equations of motion are first solved for (*u, v,w*) and finally the equation of energy is solved for the temperature. These equations are solved subject to a given initial and boundary conditions. The boundary conditions are those required by geometrical consideration together with the no-slip condition which states that, on a wall, the tangential component of relative velocity must be zero. Similarly, to solve the energy equation, some conditions must be imposed on the temperature on the boundary and these are provided depending on the nature of the problem.

3.5.1 Temperature distribution in steady, laminar incompressible fluid flow between two infinite parallel plates in Plane Couette flow

The velocity distribution for plane Couette flow has been given by equation (3.71). Hence the energy equation, for the steady flow and without heat addition, for the present problem is given by

$$\rho C_p u(\frac{\partial T}{\partial x}) = k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) + \mu(\frac{\partial u}{\partial y})^2 \qquad (3.102)$$

where k and μ are constants.

Since the velocity has been taken along the x-axis and all the variables depend on y, then $\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} = 0$. Hence, equation (3.102) reduces to

$$k = (\frac{d^2 T}{dy^2}) = -\mu(\frac{du}{dy})^2 \qquad (3.103)$$

By substituting the value of u from equation (3.71) into equation (3.103), it results into

$$\frac{d^2 T}{dy^2} = -(\frac{\mu}{k}) \times (\frac{U^2}{h^2}) \qquad (3.104)$$

Integrating equation (3.104) twice gives

$$T = -(\frac{\mu}{2k}) \times (\frac{U^2}{h^2}) y^2 + C_1 y + C_2 \qquad (3.105)$$

where C_1 and C_2 are arbitrary constants of integration.

Three cases of temperature distribution can now be considered: -

Case i: When the plates are kept at different temperatures

Suppose the lower plate ($y = 0$) and the upper plate ($y = h$) are kept at constant temperatures T_O and T_1 respectively, where $T_O > T_1$. Then for plane Couette flow the boundary conditions are:

$$T = T_O, \quad \text{when} \quad y = 0 \tag{3.106}$$

$$T = T_1, \quad \text{when} \quad y = h \tag{3.107}$$

Substituting the values of equations (3.106) and (3.107) into equation (3.105) gives.

$$T_1 = -\frac{\mu}{2k} \times \frac{U^2}{h^2} \times h^2 + C_1 h + C_2 \quad and \quad T_o = C_2 \tag{3.108}$$

Solving equation (3.108) gives

$$C_1 = \frac{T_1 - T_o}{h} + \frac{\mu U^2}{2kh} \quad and \quad C_2 = T_o \tag{3.109}$$

Substituting the values of C_1 and C_2 from equation (3.109) in to equation (3.105) results into

$$T = T_o + \frac{T_1 - T_o}{h} y + \frac{\mu U^2}{2k} \times \frac{y}{h} - \frac{\mu U^2}{2k} \times \frac{y^2}{h^2} \quad \text{or} \quad \frac{T_1 - T_o}{T_1 - T_o} = \frac{y}{h} + \frac{\mu U^2}{2k(T_1 - T_o)} \frac{y}{h}(1 - \frac{y}{h}) \tag{3.110}$$

or

$$\frac{T_1 - T_o}{T_1 - T_o} = \frac{y}{h} + \frac{1}{2} E_c \times \Pr \frac{y}{h}(1 - \frac{y}{h}) \tag{3.111}$$

where E_c =Eckert number and $Pr = Prandtl\ number$.

It follows that from equation (3.111) that the temperature distribution depends on the product $E_C \cdot Pr$. Figure 3.5 shows how dimensionless temperature $\frac{T - T_0}{T_1 - T_0}$ varies with dimensionless distance $\frac{y}{h}$ between the plates for different values of $E_C \cdot Pr$.

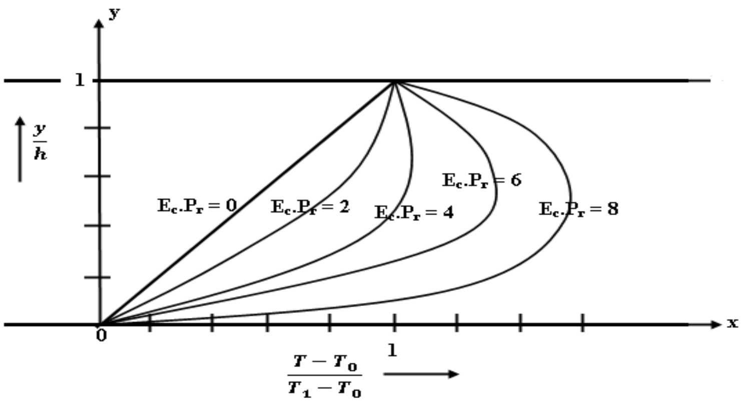

Figure 3.5 Temperature distribution for plates at different temperatures

To compute heat transfer at the upper plate, the dimensionless coefficient of heat transfer, the Nusselt Number, Nu, is described as

$$Nu = -\frac{h}{T_1 - T_o}\left(\frac{dT}{dy}\right)_{y=h} \tag{3.112}$$

Differentiating equation (3.111) with respect to y gives

$$\frac{1}{T_1 - T_o}\frac{dT}{dy} = \frac{1}{h} + \frac{1}{2}E_c \times \Pr\frac{1}{h}\left(1 - \frac{2}{h}\right) \tag{3.113}$$

Substitute $y = h$ into equation (3.113) gives

$$\left(\frac{dT}{dy}\right)_{y=h} = (T_1 - T_o)\left[\frac{1}{h} - \frac{E_c.\Pr}{2h}\right] \tag{3.114}$$

Using equation (3.114), equation (3.112) reduces to the form

$$Nu = -\frac{h}{T_1 - T_o}.(T_1 - T_o)\left[\frac{1}{h} - \frac{E_c.\Pr}{2h}\right] \tag{3.115}$$

or

$$Nu = \frac{1}{2}E_c.\Pr - 1 \tag{3.116}$$

From equation (3.141), the following observations are made-:

(i). If $E_C \cdot P_r > 2$ then N_u will be positive and the heat will be transferred from the fluid to the upper plate.

(ii). If $E_C \cdot P_r < 2$ then N_u will be negative and the heat will be transferred from the upper plate to the fluid.

(iii). If $E_C \cdot P_r = 2$ then $N_u = 0$ and there will be no heat transfer between the fluid and the upper plate.

To compute heat transfer at the lower plate,

$$Nu = \frac{h}{T_1 - T_o}\left(\frac{dT}{dy}\right)_{y=0} \tag{3.117}$$

it is shown that, at the lower stationary plate the heat is always transferred from the fluid to the plate irrespective of the range of $E_C \cdot P_r$.

Case ii: When both the plates are kept at same constant temperature (Say T_O)

The boundary conditions for this case will be:-

$$T = T_O, \qquad \text{when} \qquad y = 0 \qquad and \qquad y = h \tag{3.118}$$

Substituting these values of equation (3.118) into equation (3.105) gives

$$T_o = C_2 \; and \; T_o = -\frac{\mu}{2k} \times \frac{U^2}{h^2} \times h^2 + C_1 h + C_2 \tag{3.119}$$

By solving equation (3.119), we have

$$C_1 = \frac{\mu U^2}{2kh} \quad and \quad C_2 = T_o \tag{3.120}$$

Substituting the values of C_1 and C_2 from equation (3.120) into equation (3.105) gives

$$T - T_o = \frac{\mu U^2}{2k} \times \frac{y}{h}\left(1 - \frac{y}{h}\right) \tag{3.121}$$

This equation clearly shows that, the temperature distribution is parabolic as shown in the figure 3.6.

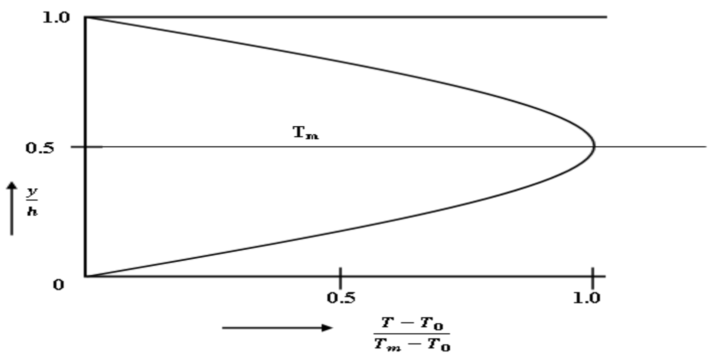

Figure 3.6 Temperature distribution for both plates kept at same constant temperature T_O

Let T_m denote the temperature distribution in the middle of the channel such that ,

$T = T_m$ when $y = \frac{h}{2}$,substituting $y = \frac{h}{2}$ in equation (3.121) gives

$$T_m - T_o = \frac{\mu U^2}{8k} \tag{3.122}$$

The Nusselt number at the lower plate for the present problem is defined as in equation (3.117).

$$Nu = -\frac{h}{(T_0 - T_m)}\left(\frac{dT}{dy}\right)_{y=0} \tag{3.123}$$

On differentiating equation (3.121) with respect to y gives

$$\frac{dT}{dy} = \frac{\mu U^2}{2k}\left(\frac{1}{h} - \frac{2y}{h^2}\right) \tag{3.124}$$

Substituting $y = 0$ in equation (3.124), we obtain

$$\left(\frac{dT}{dy}\right)_{y=0} = \frac{\mu U^2}{2kh} \tag{3.125}$$

Equation (3.123) gives

$$Nu = \frac{8kh}{\mu U^2} \times \frac{\mu U^2}{2kh} = 4$$

This shows that, the *Nusselt number for* the lower plate has a constant value 4.

Case iii: When the lower stationary plate, no heat transfer takes place (adiabatic wall) and the upper moving plate is kept at temperature T_1

The boundary conditions for the current problem are

$$\frac{dT}{dy} = 0 \quad \text{when} \quad y = 0 \tag{3.126}$$

$$T = T_1 \quad \text{when} \quad y = h \tag{3.127}$$

Integrating equation (3.104) gives

$$\frac{dT}{dy} = -\left(\frac{\mu}{k}\right) \times \left(\frac{U^2}{h^2}\right) y + C_1 \tag{3.128}$$

Putting $y = 0$ and $\frac{dT}{dy} = 0$ in equation (3.128) gives $C_1 = 0$. Next putting $y = h$ and $T = T_1$ in equation (3.105) gives

$$T_1 = -\frac{\mu}{2k} \times \left(\frac{U^2}{h^2}\right) + C_1 h + C_2 \quad \text{which implies that,} \quad C_2 = T_1 + \left(\frac{\mu U^2}{2k}\right) \quad \text{as } C_1 = 0.$$

Substituting the values of C_1 and C_2 in equation (3.105) gives

$$T - T_1 = \frac{\mu U^2}{2k}\left(1 - \frac{y^2}{h^2}\right) \tag{3.129}$$

The temperature which an insulated surface assumes under the influence of internal friction is called recovery temperature T_r. From equation (3.129) then

$$T_r = (T)_{y=0} = T_1 + \frac{\mu U^2}{2k} \quad \Rightarrow \quad T_r - T_1 = \frac{\mu U^2}{2k} \tag{3.130}$$

The recovery factor in a plane Couette flow is given by

$$r = \frac{T_r - T_1}{\frac{U^2}{2C_p}} = \frac{\mu U^2}{2k} \times \frac{2C_p}{U^2} = \frac{\mu C_p}{k} = \text{Pr} \tag{3.131}$$

Figure 3.7 illustrates temperature distribution between adiabatic wall and moving wall whose temperature is maintained at T_1.

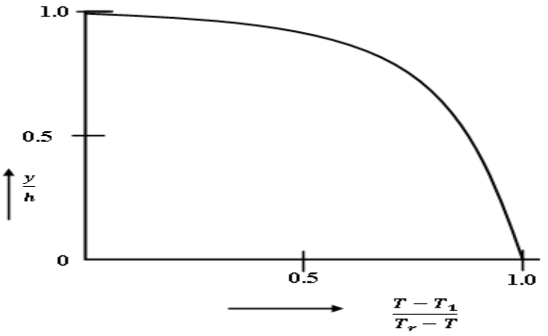

Figure 3.7 Temperature distribution between adiabatic wall and moving wall whose temp is kept T_1

3.5.2 Temperature distribution in steady, laminar incompressible fluid between two infinite parallel plates: The generalized plane Couette flow

The velocity distribution for generalized plane Couette flow from equations (3.85) to (3.86) is given as

$$\frac{u}{U} = \frac{y}{h} + \alpha\frac{y}{h}\left(1 - \frac{y}{h}\right) \tag{3.132}$$

where

$$\alpha = -\frac{h^2 P}{2\mu U} \tag{3.133}$$

is the dimensionless pressure gradient.

Using the energy equation given in equation (3.102) together with equation (3.103), equation (3.132) can be written as

$$\frac{du}{dy} = U\left(\frac{1}{h} + \frac{\alpha}{h} - \frac{2y\alpha}{h^2}\right) = \frac{U}{h}\left((1+\alpha) - \frac{2y\alpha}{h}\right) \tag{3.134}$$

On substituting the value of $\frac{du}{dy}$ into equation (3.103) gives

$$\frac{d^2T}{dy^2} = -\frac{\mu U^2}{kh^2}\left((1+\alpha)^2 - 4\alpha(1+\alpha)\frac{y}{h} + 4\alpha^2(\frac{y^2}{h})\right) \tag{3.135}$$

Integrating equation (3.135) twice with respect to y yields

$$T=-\frac{\mu U^2}{kh^2}\left((1+\alpha)^2\frac{y^2}{2}-\frac{2}{3}\alpha(1+\alpha)\frac{y^3}{h}+\frac{1}{3}\times\frac{\alpha^2 y^4}{h^2}\right)+C_1 y+C_2 \tag{3.136}$$

where C_1 and C_2 are arbitrary constants of integration to be determined. Using the boundary conditions given in equation (3.106) into equation (3.136), gives

$$T_0=C_2 \tag{3.137}$$

and

$$T_o=-\frac{\mu U^2}{kh^2}\left(\frac{1}{2}(1+\alpha)^2h^2-\frac{2}{3}\alpha(1+\alpha)h^2+\frac{1}{3}\times\alpha^2h^2\right)+C_1h+C_2 \tag{3.138}$$

on solving ,we obtain

$$C_1=\frac{\mu U^2}{kh}\left(\frac{1}{2}(1+\alpha)^2-\frac{2}{3}\alpha(1+\alpha)+\frac{1}{3}\times\alpha^2\right) \tag{3.139}$$

Substituting these values of $C_1$and C_2 into equation (3.136) gives

$$T-T_o=\frac{\mu U^2}{6k}\frac{y}{h}\left(3(1+\alpha)^2(1-\frac{y}{h})-4\alpha(1+\alpha)(1-\frac{y^2}{h^2})+2\alpha^2(1-\frac{y^3}{h^3})\right) \tag{3.140}$$

Differentiating equation (3.140) with respect to y one obtains

$$\frac{dT}{dy}=\frac{\mu U^2}{6kh}\left(3(1+\alpha)^2(1-\frac{y}{h})-4\alpha(1+\alpha)(1-\frac{y^2}{h^2})+2\alpha^2(1-\frac{y^3}{h^3})\right)+\frac{\mu U^2}{6k}\frac{y}{h}\left(-3(1+\alpha)^2\times\frac{1}{h}+4\alpha(1+\alpha)\times\frac{2y}{h^2}-2\alpha^2\right. \tag{3.141}$$

Therefore, the temperature gradient at the lower plate is given by

$$\left(\frac{dT}{dy}\right)_{y=0}=\frac{\mu U^2}{6kh}\left(2+(1-\alpha^2)\right) \tag{3.142}$$

and this shows that, heat will be always be transferred from the fluid to the lower plate irrespective of the sign of α.

3.5.3 Temperature distribution in steady, laminar incompressible fluid between two infinite parallel plates for Plane Poiseuille flow

With equations (3.72) to (3.76) in mind, the velocity distribution for the present problem is given by

$$u = -(\frac{h^2 P}{8\mu})\left(1-4\frac{y^2}{h^2}\right) \tag{3.143}$$

while the maximum velocity u_{max} is given by

$$u_{max} = -(\frac{h^2 P}{8\mu}) \tag{3.144}$$

Using equation (3.144), equation (3.143) reduces to

$$u = u_{max}\left(1-4\frac{y^2}{h^2}\right) \tag{3.145}$$

Using the energy equation for the steady flow without addition of heat given by equation (3.102) and using equation (3.103), differentiating equation (3.145) with respect to y, one gets

$$\frac{du}{dy} = u_{max}\left(\frac{-8y}{h^2}\right) \tag{3.146}$$

Hence, equation (3.103) becomes

$$\frac{d^2T}{dy^2} = -\frac{(64\mu u_{max}^2 y^2}{(kh^4)} \tag{3.147}$$

Integrating equation (3.147) twice with respect to y gives

$$T = -\frac{(16\mu u_{max}^2 y^4)}{(3kh^4)} + C_1 y + C_2 \tag{3.148}$$

where C_1 and C_2 are arbitrary constants of integration to be determined. If both the plates are kept at the same constant temperature say T_0, then for the present problem, the boundary conditions are given as

$$T = T_0 \quad \text{when} \quad y = \pm h \tag{3.149}$$

Substituting the boundary equations in equation (3.149) in equation (3.148) gives

$$T_o = -\frac{16\mu u_{\max}^2}{3k} - C_1 h + C_2 \qquad (3.150)$$

and

$$T_o = -\frac{16\mu u_{\max}^2}{3k} + C_1 h + C_2 \qquad (3.151)$$

Solving these, $C_1 = 0$ and $C_2 = T_o + \frac{16\mu}{3k} \times u_{\max}^2$. Putting this value of C_1 and C_2 into equation (3.148), one obtains

$$T - T_o = \frac{\mu u_{\max}^2}{3k}\left(1 - \frac{16 y^4}{h^4}\right) \qquad (3.152)$$

Maximum temperature T_m exists in the middle of the channel and this is obtained by substituting $y = 0$ and $T = T_m$ in equation (3.152) to obtain

$$T_m - T_o = \frac{\mu u_{\max}^2}{3k} \qquad (3.153)$$

Using equations (3.152) and (3.153) the dimensionless temperature distribution $\frac{T - T_o}{T_m - T_o}$ as a function of dimensionless distance $\frac{y}{2h}$ from the middle of the channel is given as

$$\frac{T - T_0}{T_m - T_0} = 1 - 16\left(\frac{y}{h}\right)^4 \qquad (3.154)$$

whose distribution is shown in figure 3.8.

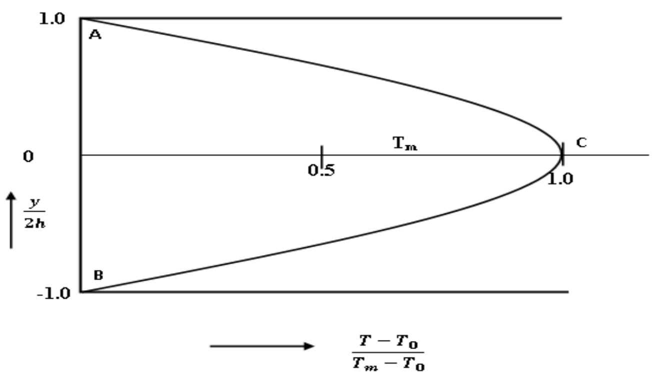

Figure 3.8 Temperature distribution for plane Poiseuille flow

3.6. Steady flow of viscous incompressible fluid between two parallel porous plates with suction/injection on the boundaries

Problems which deal with the flow of incompressible viscous fluid through porous channels and pipes are known as transpiration cooling and are known to be very effective process in reducing the heat transfer between the fluid and the boundary layer. Such problems in fluid dynamics have been found useful in cooling rockets and jets. This involves a flow in the direction, perpendicular to the main direction of the flow created by suction or injection of the fluid at the boundaries.

3.6.1 Steady viscous incompressible fluid between two infinite parallel porous plates

Consider the steady, laminar viscous incompressible fluid flow between two infinite parallel porous plates separated by a distance of $2h$. Porous plates here means that, the plates possesses very fine holes which are uniformly over the entire surface of the plates through which the fluid can continuously flow freely. The plate with injection is the plate from which the fluid enters the flow region, while the plate with suction is the plate which the fluid leaves the flow region. Let x be the direction of the main flow of the fluid, y be the direction perpendicular to the flow and the width of the plates parallel to the z direction. Taking the velocity w to be zero everywhere and u as a function of y alone, the equation of continuity then reduces to $\frac{\partial v}{\partial y} = 0$ such that v does not vary with y. Figure 3.9 shows that, the fluid enters the flow region through the plate at $y = -h$, at some constant velocity v_O, and leaves through the other plate at $y = h$ at same constant velocity v_O.

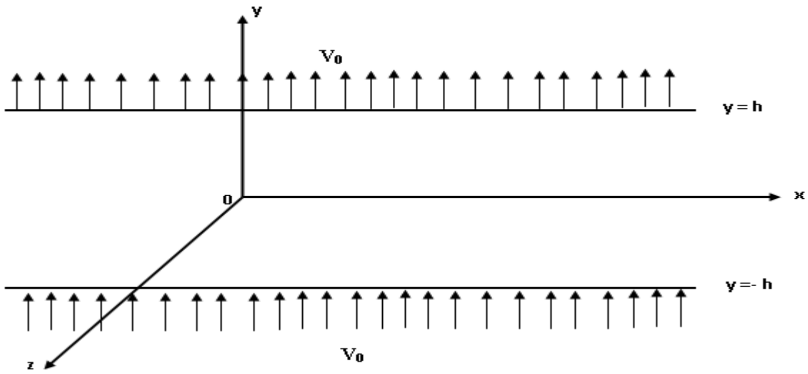

Figure 3.9 Two infinite parallel porous plates

The Navier-Stokes equation for the present problem with absence of body forces for two dimensional flow are given by

$$v_o \frac{du}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{d^2 u}{dy^2} \tag{3.155}$$

and

$$0 = -\frac{1}{\rho}\frac{\partial p}{\rho \partial y} \tag{3.156}$$

The second equation above shows that, pressure does not depend on y and hence p must be a function of x alone and therefore, equation (3.155) reduces to

$$\frac{dp}{dx} = \rho\left[\nu \frac{d^2u}{dy^2} - v_o \frac{du}{dy}\right] \tag{3.157}$$

Differentiating equation (3.157) with respect to x, we obtain $\frac{d^2 p}{dx^2} = 0$ and on integration gives

$$\frac{dp}{dx} = -P \quad (say) \tag{3.158}$$

where P is a constant and the negative sign has been taken as its expected p to decrease as x increases. With this in mind, equation (3.157) reduces to

$$\frac{d^2 p}{dy^2} - \frac{v_o}{\nu}\frac{du}{dy} = -\frac{P}{\mu} \tag{3.159}$$

Integrating equation (3.159) gives

$$\frac{du}{dy}-\frac{v_o}{v}u=C_1-\frac{Py}{\mu} \tag{3.160}$$

and this is a linear first order differential equations. The integrating factor of equation (3.160) is $e^{-\int\left(\frac{v_o}{v}\right)dy}$ and hence, the solution of equation (3.160) is given by

$$ue^{-(\frac{v_o y}{v})}=\int\left(C_1-\frac{Py}{\mu}\right)e^{-(\frac{v_o y}{v})}dy+C_2 \tag{3.161}$$

$$ue^{-(\frac{v_o y}{v})}=\left(C_1-\frac{Py}{\mu}\right)\left(\frac{v}{v_o}e^{-(\frac{v_o y}{v})}\right)-\left(-\frac{P}{\mu}\right)\left(\frac{v^2}{v_o^2}e^{-(\frac{v_o y}{v})}\right)+C_2 \tag{3.162}$$

Therefore

$$u=-\frac{v}{v_o}\left(C_1-\frac{Py}{\rho v}\right)+\frac{Pv}{\rho v_o^2}+C_2e^{\frac{v_o y}{v}} \tag{3.163}$$

or

$$u=D+\frac{P}{\rho v_o}y+Be^{\frac{v_o y}{v}} \tag{3.164}$$

where $D=-C_1\frac{v}{v_o}$, D and B are constants to be determined. Let the plate situated at $y=-h$ be at rest and the plate at $y=h$ be moving with a constant velocity U, then B and D can be determined from the following boundary conditions:

$$u=0 \quad at \quad y=-h \quad \text{and} \quad u=U, \quad at \quad y=h \tag{3.165}$$

Using the boundary conditions in equation (3.165), equation (3.164) reduces to

$$0=D-\frac{Ph}{\rho v_o}+Be^{-(\frac{v_o y}{v})} \quad and \quad U=D+\frac{Ph}{\rho v_o}+Be^{(\frac{v_o y}{v})} \tag{3.166}$$

Solving for B *and* D *for* equation (3.166) and substituting the values obtained in equation (3.164), one obtains

$$u=\left(U-\frac{2Ph}{\rho v_o}\right)\frac{e^{(\frac{v_o y}{v})}-e^{(-\frac{v_0 h}{v})}}{2\sinh(v_o\frac{h}{v})}+\frac{P}{\rho v_o}(y+h) \qquad (3.167)$$

Let Reynolds number be $\mathrm{Re}=\frac{v_o h}{v}$ and $\eta=\frac{y}{h}$, then equation (3.167) reduces to the form

$$u=\left(U-\frac{2ph^2}{\mu \mathrm{Re}}\right)\frac{e^{\eta \mathrm{Re}}-e^{-\mathrm{Re}}}{2\sinh \mathrm{Re}}+\frac{Ph^2}{\mu \mathrm{Re}}(1+\eta) \qquad (3.168)$$

which gives the velocity distribution in terms of non-dimensional quantities *Re* and η where the plates are situated at $\eta=\pm1$.

Two cases of flow either Couette flow or plane Poiseuille flow is hereby considered.

Case 1: Plane Couette flow

For plane Couette flow, there is no pressure gradient and hence equation (3.168) reduces to the form:-

$$u=\frac{1}{2}\times U(e^{\eta \mathrm{Re}}-e^{-\mathrm{Re}})\,\mathrm{csch}\,\mathrm{Re} \qquad (3.169)$$

On the other hand, the shearing stress at any point is given by

$$\alpha_{yx}=\mu\frac{du}{dy}=\frac{\mu \mathrm{Re} U e^{\eta \mathrm{Re}}}{2h\sinh \mathrm{Re}} \qquad (3.170)$$

while the skin friction at the plates $\eta=\pm1$ are given by

$$\left[\alpha_{yx}\right]_{\eta=1}=\frac{\mu \mathrm{Re} U}{2h}\frac{e^{\mathrm{Re}}}{\sinh \mathrm{Re}} \qquad (3.171)$$

and

$$\left[\alpha_{yx}\right]_{\eta=-1}=\frac{\mu \mathrm{Re} U}{2h}\frac{e^{-\mathrm{Re}}}{\sinh \mathrm{Re}} \qquad (3.172)$$

Case II: Plane Poiseuille flow

In Plane Poiseuille flow, both the plates are considered to be at stationary and the velocity distribution is obtained from equation (3.168) by letting $U = 0$ to obtain

$$u = \frac{Ph^2}{\mu \mathrm{Re}}\left(1+\eta-\frac{e^{\eta \mathrm{Re}}-e^{-\mathrm{Re}}}{2\sinh \mathrm{Re}}\right) \tag{3.173}$$

It can be verified that, the maximum velocity for the current problem occurs when

$$\eta = \frac{1}{\mathrm{Re}}\log\frac{Sinh\,\mathrm{Re}}{\mathrm{Re}} \tag{3.174}$$

and the skin friction is given as

$$\alpha_{yx} = -\frac{Ph^2}{\mathrm{Re}}\left(\frac{\mathrm{Re}\quad e^{\eta \mathrm{Re}}}{Sinh\;\mathrm{Re}}-1\right) \tag{3.175}$$

3.6.2 Steady laminar Couette flow with transpiration cooling

Consider figure 3.10, a two dimensional steady, laminar viscous incompressible fluid flow between two infinite parallel porous plates, one in uniform motion and the other at rest with uniform injection and uniform suction at the fixed plate and moving plate respectively. Let h be the distance between the plates, U the velocity of the upper moving plate and v_O the uniform injection and suction velocity at the upper plate and lower plate respectively.

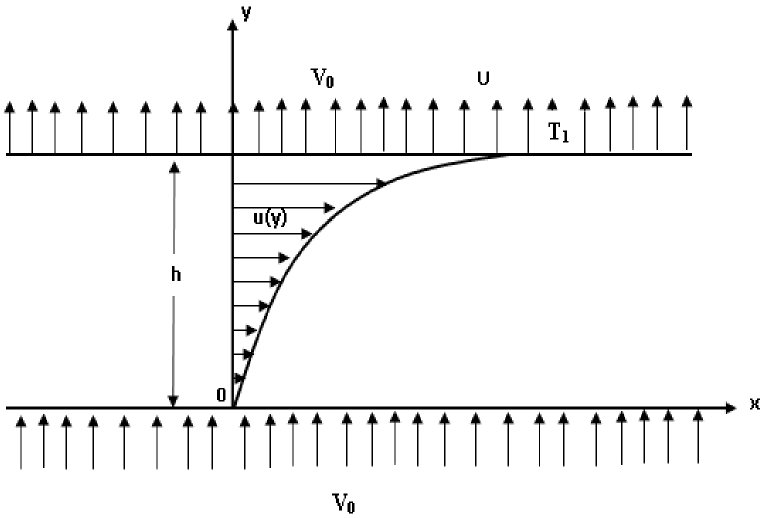

Figure 3.10 Plane Couette flow with transpiration cooling

For the present problem, the equation of continuity and equation of motion in the absence of body forces are in the form:-

$$\frac{dv}{dy} = 0 \quad (3.176)$$

$$\rho v\left(\frac{du}{dy}\right) = \mu\left(\frac{d^2u}{dy^2}\right) \quad (3.177)$$

Subject to the boundary conditions

At $y = 0$; $u = 0$, $v = v_O$; at $y = h$; $u = U$, $v = v_O$ (3.178)

and integrating equation (3.176) gives

v = constant $= v_0$ (3.179)

Substituting $v = v_O$ to equation (3.177) one obtains

$$\rho v_o\left(\frac{du}{dy}\right) = \mu\left(\frac{d^2u}{dy^2}\right) \ or \ \frac{d^2u/dy^2}{du/dy} = \frac{v_o}{v} \quad (3.180)$$

Integrating equation (3.180) twice gives

$$u = C_1 \frac{v}{v_0} \times e^{\frac{voy}{v}} + C_2 \quad (3.181)$$

where, C_1 and C_2 are constants of integration to be determined.

Using the boundary conditions (3.178), equation (3.181) reduces to

$0 = C_1 \frac{\nu}{v_o} + C_2$ and $U = C_1 \frac{\nu}{v_o} \times e^{\frac{v_{oh}}{\nu}} + C_2$ and by solving we obtain

$C_1 = (\frac{Uv_o}{\nu}) \times (e^{\frac{v_o h}{\nu}} - 1)^{-1}$ and $C_2 = -U(e^{\frac{v_o h}{\nu}} - 1)^{-1}$

Substituting the values of C_1 and C_2 in (3.181) gives

$$u = \frac{Ue^{\frac{v_o y}{\nu}}}{e^{\frac{v_o y}{\nu}} - 1} - \frac{U}{e^{\frac{v_o y}{\nu}} - 1} \quad \text{or} \quad \frac{u}{U} = \frac{e^{\frac{v_o y}{\nu}}}{e^{\frac{v_o y}{\nu}} - 1} \tag{3.182}$$

Equation (3.182) for the velocity distribution in the dimensionless form is written as

$$\frac{u}{U} = \frac{e^{\lambda\eta} - 1}{e^{\lambda} - 1} \tag{3.183}$$

where $\eta = \frac{y}{h}$ and $\lambda = v_0 \frac{h}{2}$ are defined non-dimensional quantities and λ is the injection parameter. This shows that, the velocity stops to be linear and it decreases with increasing value of λ.

3.6.3 Temperature distribution in steady laminar plane Couette flow with transpiration cooling

For the present problem, let the lower fixed and upper moving plate is possesses temperatures T_o and T_1 respectively. Let also the temperatures of the plate be constant and hence the temperature distribution in the fluid will be a function of y only. We also assume C_p, p and ρ are constants. Then the energy equation of these cases reduces to the form:-

$$\rho C_p u(\frac{dT}{dy}) = k(\frac{d^2T}{dy^2}) + \mu(\frac{du}{dy})^2 \tag{3.184}$$

The boundary conditions are :

At $y = 0$, $T = T_o$, *and at* $y = h$, $T = T_1$ (3.185)

Reducing equation (3.184) in non-dimensionless form in turn introduce four dimensionless quantities namely:- $E_c = \frac{U^2}{C_p}(T_1 - T_o)$, $\Pr = \frac{\mu C_p}{k}$, $Pe = Pr.\mathrm{Re}$ and

$T^* = \left(\frac{T - T_o}{T_1 - T_o}\right)$.

Using the above dimensionless quantities, equations (3.182) and (3.183), the energy equation (3.184) in dimensionless form reduces to the form

$$\frac{d^2T^*}{d\eta^2} - P_e\frac{dT^*}{d\eta} = -E_c\ .\ P_e\frac{e^{2\lambda\eta}}{(e^{\lambda}-1)^2}$$

or

$$(D^2 - P_e D)T^* = -E_c\, P_e\frac{e^{2\lambda\eta}}{(e^{\lambda}-1)^2} \qquad (3.186)$$

where $D = d/d\eta$. The corresponding boundary conditions are:

$$At \quad \eta = 0, \quad T^* = 0, \quad \text{at} \quad \eta = 1, \quad T^* = 1 \quad (3.187)$$

The corresponding auxiliary equation for (3.186) is $(D^2 - P_e D) = 0$, giving

$D(D - P_e) = 0$, such that, $D = 0$ or $D = P_e$.

Hence $C.F = A + Be^{Pe\eta}$, where A and B are arbitrary constants, while

$$P.I = \frac{1}{D(D - P_e)} \quad - \frac{E_c\, P_e}{(e^{\lambda}-1)^2}e^{2\lambda\eta} \quad = - \frac{E_c\, P_e}{2\lambda(2^{\lambda}-Pe)} \text{ x } \frac{e^{2\lambda\eta}}{(e^{\lambda}-1)^2}$$

. The general solution of equation (3.186) is therefore,

$$T^* = A + Be^{Pe\eta} - \frac{E_c\, P_e\, e^{2\lambda\eta}}{2\lambda(2^{\lambda}-Pe)(e^{\lambda}-1)^2} \qquad (3.188)$$

Using the boundary conditions (3.187) in (3.188), the constants A and B are obtained

which are in turn used in equation (3.188) to have

$$T^* = \frac{E_c\, P_e}{(e^{\lambda}-1)^2}\ \frac{e^{Pe\ 1-e(2\lambda-Pe)\eta}}{(2\lambda-Pe)} - \frac{e^{Pe\ 1-e(2\lambda-Pe)}}{(2\lambda-Pe)} \times \frac{e^{\eta Pe}-1}{e^{Pe}-1} + \frac{e^{\eta Pe}-1}{e^{Pe}-1} \qquad (3.189)$$

Neglecting the dissipation term, i.e. taking $E_c = 0$ (heat generated due to internal friction), then equation (3.189) reduces to

$$T^* = \frac{e^{\eta Pe}-1}{e^{Pe}-1} \qquad (3.190)$$

Nusselt Number, Nu, which is the dimensionless coefficient of heat transfer at the fixed plate, is computed as

$$Nu = -\frac{h}{(T_o - T_1)}\left(\frac{dT}{ds}\right)_{s=0} \tag{3.191}$$

Re-writing equation (3.191) in terms of T^* and η and substituting $\frac{dT^*}{d\eta}$ from equation (3.190), $Nu = \frac{Pe}{e^{Pe} - 1}$ is obtained, where if $\eta = 0$, i.e. $Pe = 0$, then one gets $Nu = 1$ and Nusselt number goes on decreasing as the value of Pe increases. This indicates cooling of the fixed plate with the injection process.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.1. Analytical method

4.1.1 Sumudu Transform

Most of the available transform theory books, if not all, rarely refer to the Sumudu Transform. Perhaps it could because no transform under this name was declared until the late 80's and early 90's of the previous century.

Sumudu Transform is an integral transform similar to the$_u$ Laplace transform which was introduced in 1993 by Gamage K. Watugala to solve differential equations and control engineering problems, (Watugala, 1993). Sumudu Transform is equivalent to Laplace - Carson Transform with the substitution of $p = \frac{1}{u}$. Hence Sumudu is a Sinhala word meaning smooth.

Sumudu Transform of a function $f(t)$ is given by the formula

$$F(u) = S[f(t);u] = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f(t)\, dt, u \in (-\tau_1, \tau_2) \qquad (4.1)$$

where the set $(-\tau_1, \tau_2)$ is the kernel of the Transform.

Watugala advocated first the transform as an alternative to the standard Laplace Transform and gave it the name Sumudu Transform. As comparable to this, Laplace transform is given as

$$\mathrm{L}f(t) = F(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad (4.2)$$

where s is a real number and L is the Laplace operator. We find that properties of Sumudu Transform are obtained from the corresponding properties of Laplace Transform.

The given theorem is useful in study of differential equations with non-constant coefficient.

Theorem 4.1. *If the Sumudu Transform of the function* $f(t)$ *is given by* $S[f(t)] = F(u)$, *then*

$$S[tf'(t)] = u^2 \frac{d}{du}\left[\frac{f(u) - f(0)}{u}\right] + u\left[\frac{f(u) - f(0)}{u}\right]$$

Proof

To proof we apply the formula given by (Asiru, 2003) given as

$$S[t\ f(t)] = u^2 \frac{d}{du}\left(f(u) + uf(u)\right)$$

This gives

$$\frac{d}{du}\left[\frac{f(u) - f(0)}{u}\right] = \frac{d}{du}\int_0^\infty \frac{1}{u} e^{\frac{-u}{t}} f'(t)dt$$

$$\frac{d}{du}\left[\frac{f(u) - f(0)}{u}\right] = \int_0^\infty \frac{d}{du}[\frac{1}{u} e^{\frac{-u}{t}}] f'(t)dt$$

$$= \frac{1}{u^3}\int_0^\infty e^{\frac{-u}{t}} f'(t)dt - \frac{1}{u^2}\int_0^\infty e^{\frac{-u}{t}} f'(t)dt$$

$$= \frac{1}{u^2} S[t\ f'(t)] - \frac{1}{u} S[f'(t)]$$

which on simplification reduces to

$$S[t\ f'(t)] = u^2 \frac{d}{du}\left[\frac{f(u) - f(0)}{u}\right] + u\left[\frac{f(u) - f(0)}{u}\right]$$

Example 1.

Find the general solution of the second order equation

$$\frac{d^2 y(t)}{dt^2} + w^2 y(t) = 0 \qquad (4.3)$$

Solution

First transform this to Sumudu equivalent:-

$$\frac{G(u) - y(0)}{u^2} - \frac{y'(0)}{u} + w^2 G(u) = 0 \qquad (4.4)$$

$$\frac{G(u) - y(0) + uy'(0) + u^2 w^2 G(u)}{u^2} = 0 \qquad (4.5)$$

$$G(u)[1 + u^2 w^2] = y(0) + uy'(0) \qquad (4.6)$$

which gives the general solution as

$$G(u) = \frac{y(0) + uy'(0)}{1 + u^2 w^2} \tag{4.7}$$

and upon inverting the general time solution is given as

$$y(t) = y(0)\cos(wt) + \frac{y'(0)}{w}\sin(wt) \tag{4.8}$$

Example 2

Solve a differential equation

$$\frac{d^2 y(t)}{dt^2} - M^2 y(t) = 0 \tag{4.9}$$

by the method of Sumudu Transform.

Solution

Transform this into its Sumudu equivalent as-:

$$\frac{G(u) - y(0)}{u^2} - \frac{y'(0)}{u} - M^2 G(u) = 0 \tag{4.10}$$

On rearranging, the general solution is

$$G(u) = \frac{y(0) + uy'(0)}{1 - M^2 u^2} \tag{4.11}$$

and upon inverting the general time solution is

$$y(t) = y(0)\cosh(Mt) + \frac{y'(0)}{M}\sinh(Mt) \tag{4.12}$$

4.1.2 Sumudu Transform on Steady Magnetohydrodynamic Couette Flow between Two Infinite Parallel Plates

Consider an electrically conducting, laminar, viscous, steady and incompressible fluid moving between two infinite parallel plates both kept at a constant distance h between them. The upper plate is moving with constant velocity U_O while the lower plate is kept stationary under the influence of uniform transverse magnetic field $\mathbf{B}_O$. The fluid is acted upon by a constant pressure gradient.

In describing MHD phenomena on steady Couette flow between two infinite parallel plates, consider the conceptual framework of MHD from figure 1.1 and equations (1.1) and (1.2).

The Lorentz force

$$\boldsymbol{F}_{in} = \mathbf{J}_{in} \times \boldsymbol{B}_{app} \qquad (4.13)$$

is important here in determining the flow profile based on the dimensionless Hartmann number which is given $Ha = \sqrt{(N \cdot Re}$, where $N = \frac{\sigma L B^2}{\rho} U$ stands for non dimensional interaction parameter known as Stuart number which is defined as the ratio of electromagnetic to inertial forces, and this gives an estimate of the relative importance of a magnetic field of the flow. It is also relevant for flows of conducting fields e.g. in fusion reactors, steel casters or plasmas. On the other hand, $Re = \frac{UL}{\nu}$ is the non dimensional hydrodynamic Reynolds number, so the Hartmann number can be rewritten as $Ha = LB\sqrt{\frac{\sigma}{\mu}}$ where μ is the dynamic viscosity and υ is the kinematic viscosity.

The governing equations for the flow of incompressible Newtonian fluid that we use in this study are the continuity equation and the momentum equations which are given as:

$$\nabla \cdot \mathrm{V}=0 \qquad (4.14)$$

$$\rho\left[\frac{\partial \mathrm{V}}{\partial t}+(\mathrm{V}.\nabla)\mathrm{V}\right]=-\nabla p+\mu\nabla^2\mathrm{V}+\mathrm{J}\times\mathrm{B} \qquad (4.15)$$

where ρ is the fluid density, p is the fluid pressure function acting on the fluid, μ is the fluid dynamic viscosity and $\mathbf{J} \times \mathbf{B}$ is the Lorentz force.

We now non-dimensionalize the governing equations. Consider unidirectional flow, in which we choose the axis of the channel formed by the two plates as the x-axis and assume that the flow is in this direction. If $\mathrm{V} = u'(x',y',z')\hat{i}+v'(x',y',z')\hat{j}+w'(x',y',z')\hat{k}$ in which u', v' and w' are the components of the velocity in x, y and z directions respectively and primes denote dimensionless quantities. This now implies that $v'=w'=0$ and that $u'\neq 0$. Then the continuity equation (4.14) yields $\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}=0$. But $\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}=0$ so that $\frac{\partial u'}{\partial x'}\neq 0$ from which we infer that u' is independent of x' .

This makes the non- linear term $(V.\nabla)V$ in the Navier-Stokes equations equal to zero because of the unidirectional flow assumption. Since we had assumed a steady flow, the flow variable does not depend on time. By assuming that the flow is two dimensional, i.e. the flow variables are independent of z- direction, then this means that, by choosing the axis of the channel as the x- axis, the governing equations of motion for two dimensional steady flow are:-

$$0 = -\frac{1}{\rho}\frac{\partial p'}{\partial x'} + \nu\frac{\partial^2 u'}{\partial y'^2} + \frac{F_x}{\rho} \tag{4.16}$$

$$0 = -\frac{1}{\rho}\frac{\partial p'}{\partial y'} \tag{4.17}$$

F_X is the component of the magnetic force in the x-direction. We note that p is a function of x only and that pressure does not depend on y from equation (4.17) . Also, assuming unidirectional flow $v' = w' = 0$ and $\mathbf{B}_X = \mathbf{B}_Z = 0$ so that $\mathbf{V} = u'\hat{i}$ and $\mathbf{B} = \mathbf{B_0}\,\hat{j}$ where $\mathbf{B}_0$ is the magnetic field strength component assumed to be applied to a direction perpendicular to fluid motion (y-direction). Therefore, using the cross product of (4.16) we obtain

$$F_x = \sigma[(u'\hat{i} \times B_o\hat{j}) \times B_o j'] \tag{4.18}$$

which reduces

$$F_x = -\frac{\sigma}{\rho}B_o^2 u' \tag{4.19}$$

Substituting equation (4.19) into equation (4.16) gives

$$0 = -\frac{1}{\rho}\frac{\partial p'}{\partial x'} + \nu\frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma}{\rho}B_o^2 u' \tag{4.20}$$

or

$$\frac{d^2 u'}{dy'^2} - \frac{\sigma}{\rho}B_o^2 \sin(\alpha)u' = \frac{1}{\mu}\frac{dp'}{dx'} \tag{4.21}$$

where we have taken ordinary derivatives instead of partial derivatives and α is the angle between **V** and **B**. Equation (4.21) is a general equation in that the two fields can be assessed at any angle for $0 \le \alpha \le \pi$ and is solved subject to boundary conditions:-

$$u = 0, \quad y = -1 \quad u = U_O, \quad y = +1 \tag{4.22}$$

Dropping the primes in equations (4.21) and (4.17) for convenience,

$$\frac{d^2u}{dy^2} - \frac{\sigma}{\rho} B_o^2 \sin(\alpha) u = \frac{1}{\mu}\frac{dp}{dx} \tag{4.23}$$

$$0 = -\frac{1}{\rho}\frac{dp}{dy} \tag{4.24}$$

Since the flow is Couette then pressure gradient is taken to be zero i.e. $\frac{dp}{dx} = 0$ in equation (4.23). If we let l be the characteristic length, the dimensionless equation (4.23) reverts back to the non-dimensionless form and define the dimensionless quantities as

$$x = \frac{x'}{l}, \quad y = \frac{y'}{l}, \quad p = \frac{p' l^2}{\rho \nu^2}, \quad u = \frac{u' l}{\nu} \tag{4.25}$$

Substituting the quantities of equation (4.25) into equation (4.23) gives

$$\frac{d^2u}{dy^2} - \frac{\sigma}{\mu} B_o^2 \sin^2(\alpha) u = 0 \tag{4.26}$$

or

$$\frac{d^2u}{dy^2} - M^2 u = 0 \tag{4.27}$$

where $M = M^* \sin\alpha$ is the Hartmann number.

Solving the second order differential equation (4.27) by using Sumudu Transform method and by first evaluating initial conditions with the help of boundary conditions given in equation (4.22). Equation (4.27) can be transformed to its Sumudu equivalent as

$$\frac{G(u) - y(0)}{u^2} - \frac{y'(0)}{u} + M^2 G(u) = 0 \tag{4.28}$$

Multiplying all through by u^2 and on rearranging gives

$$G(u) - y(0) - u y'(0) - u^2 M^2 G(u) = 0 \tag{4.29}$$

or

$$G(u)[1 - M^2 u^2] = y(0) + u y'(0) \tag{4.30}$$

Let $y(0) = c_1$ and $y'(0) = c_2$, then equation (4.30) reduces to

$$G(u) = \frac{c_1 + c_2 u}{1 - M^2 u} \tag{4.31}$$

Upon inverting, the general solution is given as

$$u(y) = c_1 \cosh M - c_2 \sinh M \tag{4.32}$$

Using the boundary conditions given in equation (4.22), gives

$$0 = c_1 \cosh M - c_2 \sinh M \tag{4.33}$$

$$U_0 = c_1 \cosh M + c_2 \sinh M \tag{4.34}$$

On solving equations (4.33) and (4.34), $c_1 = \dfrac{U_0}{2\cosh M}$ and $c_2 = \dfrac{U_0}{\sinh M}$ and substituting these into equation (4.32) gives

$$u(y) = \frac{U_0}{2}\left[\frac{\cosh My}{\cosh M} - \frac{\sinh My}{\sinh M}\right] \tag{4.35}$$

or

$$\frac{u(y)}{U_0} = \frac{\sinh[M(1+y)]}{\sinh 2M} \tag{4.36}$$

is obtained.

Equation (4.36) has been solved using Sumudu Transform for the linear differential equation with constant coefficients. The constants c_1 and c_2 obtained shows that, the expression for the velocity of fluid particles can be derived in terms of hyperbolic functions. Equation (4.36) is then used to draw velocity profiles for various Hartmann number and for various angles of inclination.

Figure 4.1 shows velocity profiles for Hartmann number $M^* = 0.5$, $M^* = 1.5$ and $M^* = 2.5$ for $\alpha = 15^o$ as the angle of inclination.

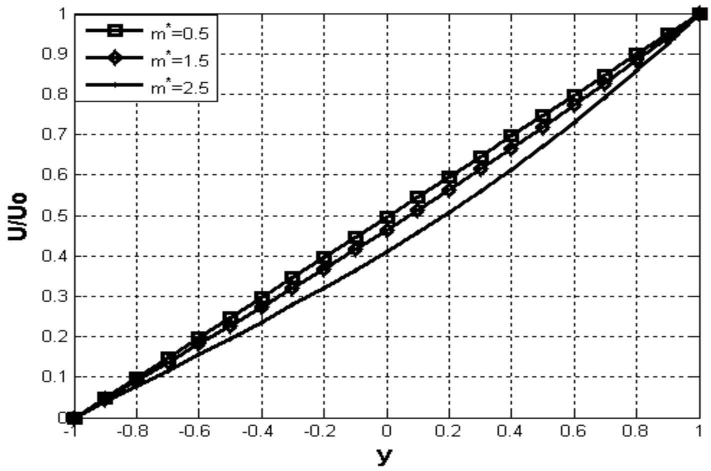

Figure 4.1 Couette flow velocity profile for various Hartmann number and $\alpha = 15^O$

Figure 4.2 shows velocity profiles for Hartmann number $M^* = 0.5$, $M^* = 1.5$ and $M^* = 2.5$ for $\alpha = 30^O$ as the angle of inclination.

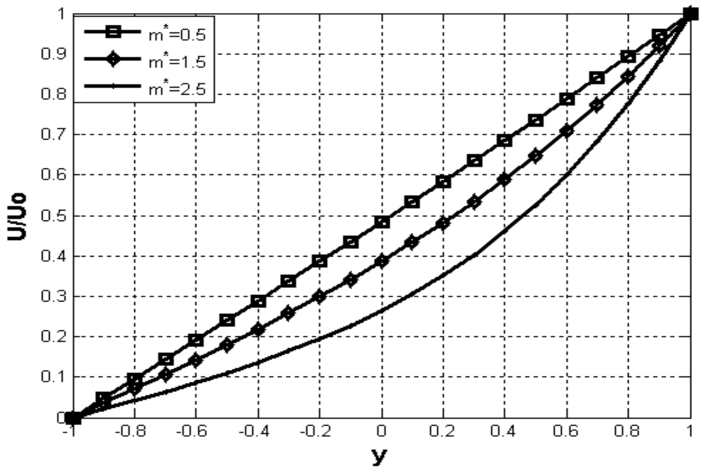

Figure 4.2 Couette flow velocity profile for various Hartmann number and $\alpha = 30^O$

Figure 4.3 shows velocity profiles for Hartmann number $M^* = 0.5$, $M^* = 1.5$ and $M^* = 2.5$ for $\alpha = 45^O$ as the angle of inclination.

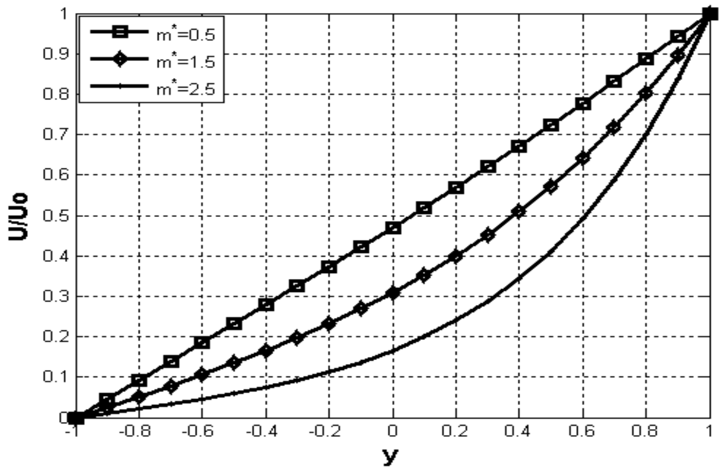

Figure 4.3 Couette flow velocity profile for various Hartmann number and $\alpha = 45^o$

Figures 4.1 to 4.3 has been drawn for Hartmann numbers $M^* = 0.5$, $M^* = 1.5$ and $M^* = 2.5$ and angle of inclinations of $\alpha = 15^o$, $\alpha = 30^o$ and $\alpha = 45^o$. The three figures show that, increase in magnetic field strength and magnetic inclinations results into decreases in the Couette flow velocity profiles.

4.2. Numerical methods

Many real-life problems modeled mathematically do not have analytical solutions. Hence, MHD being a scientific research area of study leads to real life problems requiring the use of numerical techniques to accomplish non-analytical solutions. Second order partial differential equations govern many of the real-life physical phenomena of such nature. The numerical analysis part which has been most changed so far, is the solution of partial differential equations by difference methods. A general and a powerful method of dealing with such kind of second order partial differential equations is the finite difference method.

4.2.1 Finite difference method

This method uses finite difference codes/solvers that take low computational memory and is easy to program and modify. It involves the study of (time-dependent) partial differential equations, whose solutions vary both in time and in space. There are a lot of interesting problems that cannot be solved by analytical methods. It is thus advantageous to use finite difference method in MHD problems. Equations arising in MHD include Maxwell's field equations, heat and momentum equations and the Newton's law of motion if coupled with Navier-Stokes equations. Three important properties of finite difference equations must be considered for computational purposes, namely;

•*Consistency:* A difference equation is said to be consistent or compatible with partial differential equation when its truncation error approaches zero. This is equivalent to Truncation error $\rightarrow 0$ as mesh size $\rightarrow 0$.

•*Stability:* A numerical scheme is stable if errors from any source (truncation errors and round off errors in measurements) are not permitted to grow as the calculation proceeds or magnified at each iteration. The problem of stability is very important in numerical analysis. There are two methods for checking the stability of linear difference equations. The first one is referred to as Fourier approach or von Neumann which assumes the boundary conditions are periodic. The Fourier method is based on decomposing the numerical solutions into Fourier harmonics on the spatial grid. Although this method does not capture the influence of boundary conditions, it is easy to formulate and usually accurate enough to provide practical stability criteria. The second one is called the matrix method which takes care of contributions to the error from the boundary.

•*Convergence:* A scheme is said to be convergent if the solution to the finite difference equation approaches the exact solution to the partial differential equation with the same initial and boundary conditions as the mesh size approaches zero. Lax on his equivalence theorem showed that, under appropriate conditions a consistent scheme is convergent if and only if it is stable. It states that, " given a properly posed linear initial value problem and a finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence" (Richtmeyer and Morton, 1967).

The issue of accuracy of a numerical scheme is not very relevant on its own right. A consistent scheme can be made increasingly accurate by decreasing the time and spatial steps. What matters is the cost (coding effort, memory requirements and computational requirements) of the accuracy.

The use of the finite difference techniques for the solution of partial differential equations has three step process namely:

1. The partial differential equations are approximated by a set of linear equations relating to the values of the functions at each mesh point.
2. The set of the algebraic equations, generated in the first must be solved.
3. An iteration procedure has to be developed which takes into account the non- linear characters of the equation.

Solutions of the finite difference equation (FDE) requires suitable techniques to advance the transient fluid motion through time. The transient terms in the equations are dropped and the problem is simplified and reduced to just determining the steady state solution if the transient solutions are not required. In this study, the partial differential equations governing the flow are replaced by a set of difference equations and the governing equations together with initial and boundary conditions imposed (depending on the problem considered) are properly posed, (i.e. their solution exists, is unique and depends on the given conditions) thus any finite difference set of equations to them which satisfies consistency conditions and is stable ensure that the method is convergent. In order to solve the system of finite difference equations, a computer program will be used for the iterative scheme. To approximate the differential equation by a set of finite difference equations we first require to define a suitable mesh. One of the first steps in using finite difference methods is to replace the continuous problem domain by a difference mesh or a grid. In the finite-difference method, the partial derivatives in the governing equations are discretized directly by solving the equations on a discrete set of points.

4.2.2 Discretization techniques

To solve partial differential equations, one must discretize the partial derivatives. To give an explicit relation between the partial derivatives in equations and the function values at the adjacent nodal points, a uniform mesh which involves subdividing the rectangular region of interest into uniform rectangular elements is used, centered about mesh point whose coordinates are denoted by integer variables. For the steady non-linear coupled ordinary differential equations with initial and boundary conditions, they will be solved by employing the centered finite difference scheme. The discretization will provide a useful and consistent approximation to the solutions in dimensionless governing equations.

Let the solution in 2-D of the differential equation be designated as $u(x, y)$, where the two independent variables are x and y. On specifying each of the independent variables as grid as

$$x_{i=x_0} + i\Delta x \qquad i = 0, \dots\dots\dots\dots N_x$$

$$y_{i=x_0} + i\Delta y \qquad i = 0, \dots\dots\dots\dots N_y$$

$$x_{ij} \equiv u(x_i, y_j$$

the partial derivatives of u can be approximated by finite difference expressions in the x and y directions.

Forward difference

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x)$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O(\Delta x)$$

Central difference

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x^2)$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O(\Delta x^2)$$

From above, the forward difference equations are accurate only to first order in the step size Δx, whereas the central difference equations are accurate to second order in the step size Δx.

For a function of two variables say x and t the definition of partial derivatives with respect to time t and with respect to x respectively are given as

$$\frac{\partial u}{\partial t}(x,t) = \lim_{\Delta t \to 0} \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}|_{x\ fixed}$$

$$\frac{\partial u}{\partial x}(x,t) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}|_{t\ fixed}$$

and assuming that there are two independent variables y and t then we simply relabel t and y and everything stated above will remain true. For discretization purposes, the standard notation for finite difference by letting $\Delta x = h$ and $\Delta t = k$ for a function of two variables x and t is used. These step sizes will then be used to discretize the continuous time and space intervals:

$$[0, x_{max}] \to [x_i, - - - - - - - - -, x_n]$$

and

$$[0, t_{max}] \to [t_i, - - - - - - - - -, t_m]$$

where

$$x_{n+1} = x_n + h, \quad \forall n = 1, -0 - - - - - - -, n-1$$

and

$$tx_{m+1} = t_{nm} + k, \qquad \forall m = 1, -0 - - - - - - -, m-1$$

For a function of two spatial variables and time variable $u = u(x, y, t)$, then

$$\frac{\partial u}{\partial t}(x,y,t) = \lim_{\Delta t \to 0} \frac{u(x,y,t+\Delta t) - u(x,y,t)}{\Delta t}|_{x,y\ \ fixed}$$

$$\frac{\partial u}{\partial x}(x,y,t) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x,y,t) - u(x,y,t)}{\Delta x}|_{y,t\ fixed}$$

$$\frac{\partial u}{\partial y}(x,y,t) = \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y,t) - u(x,y,t)}{\Delta y}|_{x,t\ fixed}$$

Let $\Delta x = h$, $\Delta t = k$ or $\Delta x = h$ and $\Delta y = k$ for the case of 2-dimensional, using the forward difference in time step to approximate u_t and a centered difference in space step and time step to approximate u_{xx} , u_{tt} and u_{yy} yield the expression:-

Centered difference in time $\quad \frac{\partial u}{\partial t}(x_n, t_m) = \frac{u_{n,m+1} - u_{n,m}}{k} + O(k)$

Centered difference in space $\quad \frac{\partial^2 u}{\partial x^2}(x_n, t_m) = \frac{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}}{h^2} + O(h^2)$

Forward difference in time $\frac{\partial^2 u}{\partial t^2}(x_n,t_m) = \frac{u_{n,m+1} - 2u_{n,m} + u_{n,m-1}}{k^2} + O(k^2)$

Centered difference in space $\frac{\partial^2 u}{\partial y^2}(x_n,y_m) = \frac{u_{n,m+1} - 2u_{n,m} + u_{n,m-1}}{k^2} + O(k^2)$

4.2.3 Thomas algorithm

The Thomas algorithm, (Douglas & Randall, 1999) is an efficient way of solving tri-diagonal matrix systems. If L is a lower triangular matrix and U is an upper triangular matrix, Thomas algorithm is based on LU decomposition in which the matrix system $Mx = r$ is rewritten as $LUx = r$. The system can be efficiently solved by setting $Ux = \rho$ and then solving first $L\rho = r$ for ρ and then $Ux = \rho$ for x. The Thomas algorithm consists of two steps. Step one involves decomposing the matrix into $M = LU$ *and* solving $L\rho = r$ and this is accomplished in a single downwards sweep, taking us straight from $Mx = r$ to $Ux = \rho$. In the second step, the equation $Ux = \rho$ is solved for x in an upwards sweep.

Thomas algorithm is used because it is fast and because tri-diagonal matrices often occur in practice. The condition for the algorithm to be stable is $\|b_i\| > \|a_i\| + \|c_i\|$ for all i. The matrix problems which result from the discretisation of partial differential equations nearly all satisfy this criterion. If the algorithm is numerically unstable then one must rearrange the equations by pivoting. Standard LU decomposition algorithms for full or banded matrices include pivoting, though one has to first check to make sure no mistake has been made when formulating the problem.

The Thomas' algorithm MATLAB code used in this work is given as:-

```
function x = thomas (n, a. b. c. d)
c(1) = c(1)/b(1) ;
d(1) = d(1)/b(1) ;
for i = 2: n
b(i) = b(i) − a(i) ∗ c(i − 1);
```

$$c(i) = c(i)/b(i);$$

$$d(i) = (d(i)\text{-}d(i\text{-}1) * a(i))/b(i);$$

$$end;$$

$$x(n) = d(n);$$

$$for\ k = n - 1: -1: 1$$

$$x(k) = d(k) - c(k) * x(k + 1);$$

$$end;$$

$$x = x';$$

4.2.4 Numerical methods on steady MHD Poiseuille fluid flow between two infinite parallel porous plates

Consider laminar electrically conducting viscous, steady, incompressible fluid moving between two infinite parallel plates both of which are kept at a constant distance $2h$ between them. The upper plate and the lower plate are kept stationary. The fluid is acted upon by a constant pressure gradient which makes this flow a plane Poiseuille flow. Let the laminar MHD steady incompressible fluid between the two infinite parallel porous plates be under the action of an external uniform transverse magnetic field applied transverse to the flow direction. Both the lower plate and the upper plates are assumed porous and the fluid enters the flow region through the lower plate and leaves through the upper plate with constant velocity v_O.

For zero displacement and Hall currents, Maxwell's equations together with Ohms law and Law of magnetic conservation are given in equations (1.21) to (1.24).

The governing equations for the flow of incompressible Newtonian fluid used in this study are the continuity equation and the momentum equations which are given in equations (4.14) and (4.15).

For the present problem, the following assumptions are made:

1. The fluid flow is incompressible,
2. the fluid flow is steady hence the flow variables do not depend on time,
3. the fluid is electrically neutral i.e. there is no surplus electrical charge

distribution present in the fluid,

4. the only body forces are Lorentz forces,
5. the fluid flow is unidirectional in x - axis, the channel formed by the two plates,
6. the flow is laminar i.e. the flow paths of individual particles of the fluid do not cross those of neighboring particles, hence, making it possible to follow the path/motion of every individual particle.

The equation of continuity and the momentum equations in two dimensions together with the above assumptions reduce equations (4.14) and (4.15) to the form:-

$$\frac{\partial v}{\partial y} = 0 \tag{4.37}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\frac{\partial^2 u}{\partial y^2} + \frac{F_x}{\rho} \tag{4.38}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{4.39}$$

From equation (4.37), it implies that v = constant or v = 0 and hence, flow geometry implies that v cannot be a constant and therefore, we choose $v = 0$. Pressure does not depend on y from equation (4.39), Hence, p is a function of x alone.

Equations (4.38) can be non-dimensionalized using the characteristic velocity U, the body length L by denoting the dimensional quantities given as

$$x = \frac{\tilde{x}}{L},\ y = \frac{\tilde{y}}{L},\ p = \frac{\tilde{p}L^2}{\rho v^2},\ u = \frac{\tilde{u}L}{v} \tag{4.40}$$

And subsequently solving subject to boundary conditions $\tilde{u} = 0$ when $\tilde{y} = \pm L$, where bars denote dimensionless quantities.

Using assumptions (3), (4) and (5) $\mathbf{B}_z$ = $\mathbf{B}_x$ = 0 and $\tilde{u} = \tilde{w} = 0$ so that, $v_x = \tilde{u}\,\hat{i}$ and $B = B_0\hat{j}$ where B_o is the magnetic field strength component assumed to be applied to a direction perpendicular to the fluid motion in $\tilde{y}$-direction, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors in the x and y - directions respectively. Note that, $F_x = \sigma[\tilde{u}\hat{i} \times B_o\hat{j}] \times B_o\hat{j}$, from which $\frac{F_x}{\rho} = -\frac{\sigma}{\rho} B_o^2 \tilde{u}$. Using the dimensionless quantities given in equation (4.30), equation (4.38) reduces to

$$0 = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \nu \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} - \frac{\sigma}{\rho} B_0^2 \tilde{u} \tag{4.41}$$

or

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} - \frac{\sigma}{\mu} B_0^2 \tilde{u} = \frac{1}{\mu}\frac{\partial \tilde{p}}{\partial \tilde{x}} \tag{4.42}$$

or

$$\frac{d^2 u}{dy^2} - \frac{\sigma}{\mu} B_0^2 (\sin\alpha) u = \frac{1}{\mu}\frac{dp}{dx} \tag{4.43}$$

where ordinary derivatives instead of partial derivatives has been taken and α is the angle between **V** and **B** which means that, the two fields can be assessed at any angle α for $0 \le \alpha \le \pi$.

Differentiating equation (4.43) with respect to *x*, $\frac{d^2 p}{dx^2} = 0$ is obtained and on integration this $\frac{dp}{dx} = -c$ (a constant) is obtained. Dropping the bars (for convenience) we get $\frac{d^2 u}{dy^2} - \frac{\sigma}{\rho} B_o^2 \sin(\alpha) u = \frac{1}{\mu}\frac{dp}{dx}$

or

$$\frac{d^2 u}{dy^2} - M^2 u - \frac{1}{\mu}\frac{d\tilde{p}}{dx} = 0 \tag{4.44}$$

where $M = M^* \sin\alpha$ is the Hartmann number.

Equation (4.44) can be rewritten as

$$\frac{d^2 u}{dy^2} - M^2 u + C = 0 \tag{4.45}$$

whose solution subject to boundary conditions $u = 0$,when $y = \pm 1$ is

$$\frac{u}{C} = \frac{1}{M^2}\left[1 - \frac{\cosh My}{\cos M}\right] \tag{4.46}$$

by method of solution of differential equation with constant coefficients.

Now consider the MHD steady, laminar flow of viscous incompressible fluid between two infinite parallel porous plates separated by a distance $2h$ and x- axis be taken in the middle of the channel parallel to the direction of flow, the y direction perpendicular to the flow, and the width of the plates parallel to the z – direction. The word infinite here means that the width of the plates is large compared with h and hence we treat the flow to be two dimensional. Take also the velocity component w to be zero everywhere and u as function of y alone.

Since both plates have very fine holes distributed uniformly over the entire surface of the plates through which the fluid can flow freely and continuously, the fluid will enter the flow region through the lower plate and leave through the upper plate with constant characteristic velocity v_o along y -direction. For the present steady flow the equation of continuity reduces to $\frac{du}{dy}=0$, so that v does not vary with y. Similarly, the x and y momentum equations are given by

$$v_o\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{4.47}$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{4.48}$$

Equation (4.48) shows that pressure does not depend on y and therefore the equation collapses as p is a function of x alone and so equation (4.47) reduces to

$$\frac{dp}{dx}=\rho\left[\nu\frac{d^2u}{dy^2}-v_o\frac{du}{dy}\right] \tag{4.49}$$

Differentiating equation (4.49) with respect to x , we obtain $\frac{d^2p}{dx^2}=0$ or $\frac{d}{dx}(\frac{dp}{dx})=0$.

Integrating, $\frac{dp}{dx}=-P$ (a constant-say), where the negative sign has been taken to show pressure decreases as x increases. Substituting this, equation (4.49) now becomes

$$\frac{d^2u}{dy^2}-\frac{v_o}{\nu}\frac{du}{dy}=-\frac{P}{\rho\nu} \tag{4.50}$$

If the fluid is subjected to uniform transverse magnetic forces, equation (4.50) can be modeled by adding the term $(-M^2u)$ to yield

$$\frac{d^2u}{dy^2} - \frac{v_o}{\nu}\frac{du}{dy} + \frac{P}{\mu} - M^2u = 0 \tag{4.51}$$

`Let equation (4.51) be of the form

$$\frac{d^2u}{dy^2} - A\frac{du}{dy} - M^2u + B = 0 \tag{4.52}$$

where $A = \frac{v_o}{\nu}$ and $B = \frac{P}{\mu}$ are arbitrary constants of the fluid to be determined.

The differential equation (4.52) with initial and boundary conditions

$$u = 0; \quad y = \pm 1 \tag{4.53}$$

is to be solved by using finite difference approach. In this method, derivatives occurring in the generated differential equations are replaced by their finite differences. The resultant linear equation has been solved by central difference approximation. Central difference approximations are used because they are more accurate than forward and backward differences.

The numerical computation of the generated linearized system of equations based on the step size and results of these are achieved with the aid of MATLAB application software. Representing the step size by k, the finite difference equation corresponding to equation (4.52) is given as

$$\frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} - A\frac{U_{i,j+1} - U_{i,j-1}}{2k} - M^2U_{i,j} + B = 0 \tag{4.54}$$

where i and j are for number Δt and Δy increments respectively. The block tri- diagonal system is solved using Thomas algorithm. All calculations have been carried out for $A = 1$, $B = 2$ and $k = 0.25$. Figure 4.4 shows Poiseuille velocity profiles for Hartmann number $M^* = 0.5$, $M^* = 1.5$ and $M^* = 2.5$ for $\alpha = 15^o$ as the angle of inclination.

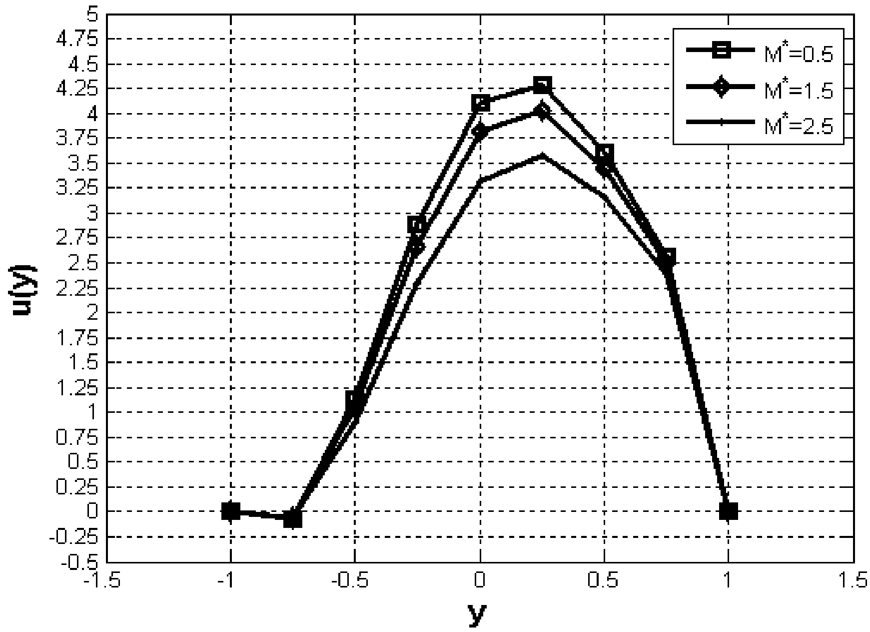

Figure 4.4. Poiseuille flow velocity profile for various Hartmann number and $\alpha = 15^{o}$

Figure 4.5 shqows Poiseuille velocity profiles for Hartmann number $M^* = 0.5$, $M^* =$ 1.5 and M^* =2.5 for $\alpha =$ 30^{o} as the angle of inclination.

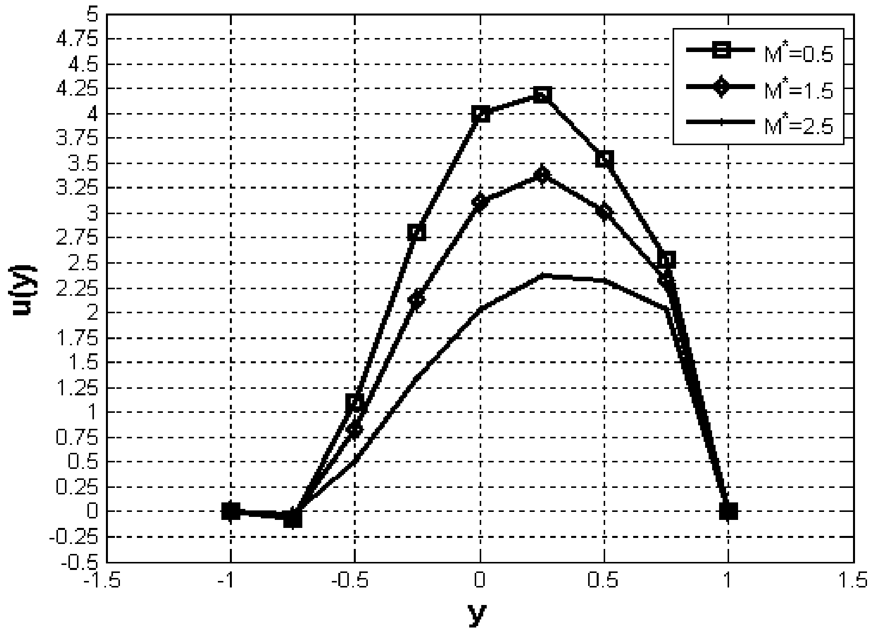

Figure 4.5 Poiseuille flow velocity profile for various Hartmann number and $\alpha = 30^{o}$

Figure 4.6 shows Poiseuille velocity profiles for Hartmann number $M^* = 0.5$, $M^* = 1.5$ and $M^* = 2.5$ for $\alpha = 45^{o}$ as the angle of inclination.

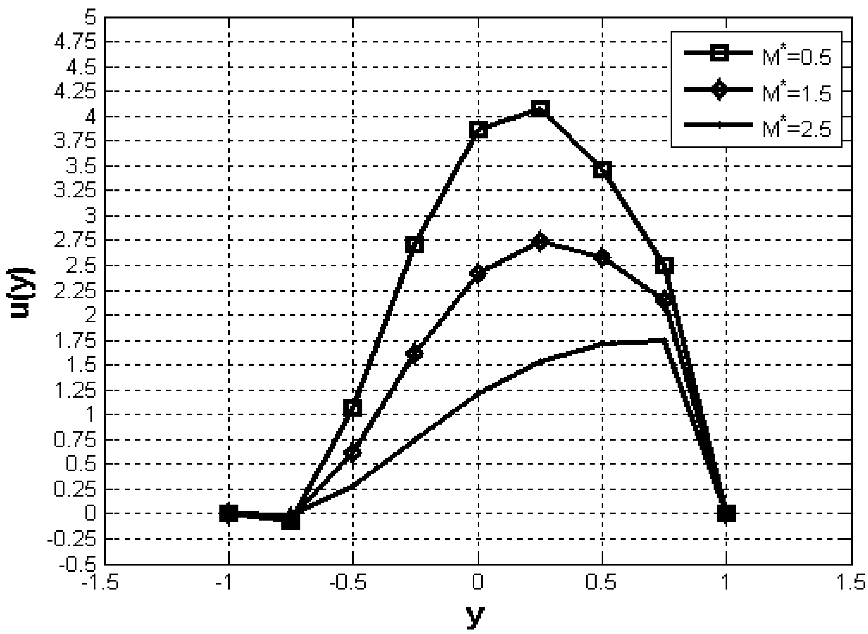

Figure 4.6 Poiseuille flow velocity profile for various Hartmann number and $\alpha = 45^0$

Figures 4.4 to 4.6 show how numerical calculations have been performed for velocity profiles in Poiseuille flow. The results are presented graphically for various Hartmann numbers and different angles of inclinations. The results show how velocity of the fluid changes with varied Hartmann numbers. An increase in the Hartmann number from $M^* = 0.5$ to $M^* = 2.5$ leads to a decrease in velocity distribution in Poiseuille flow. This is due to Lorentz force generated by the application of constant transverse magnetic field which offers resistance, hence, opposing the fluid motion and hence decreasing the flow.

4.2.5 Numerical methods on unsteady incompressible MHD Couette flow with heat transfer between two parallel porous plates

The present research obtains the results of a better understanding of the basic heat transfer associated with continuously moving boundary under the influence of applied magnetic field and providing insight into process in the systems related to manufacturing applications. Problems of fluid flow and heat transfer involving boundary of fluid in a passage can be found in many manufacturing applications.

In the present problem, unsteady MHD Couette laminar flow of viscous incompressible fluid between two infinite parallel porous plates in presence of uniform magnetic field is studied. The upper and lower plates are maintained at two different but constant temperatures T_2 and T_1 respectively, with $T_2 > T_1$. The upper plate is moving with constant velocity U_o while the lower plate is kept stationary. A sudden uniform and a constant pressure gradient, an external uniform magnetic field with magnetic flux density vector $\mathbf{B}_o$ is applied in the positive y- direction which is assumed to be also the total magnetic field. The flow is subjected to a uniform suction from above and uniform injection from below at $t = 0$. The two parallel non-conducting plates are at a distance $2h$ apart and the flow is in the x -axis direction through the central line of the channel and y axis normal to it. This means that, the plates of the channels are at $y = \pm h$ and the relative velocity between the two plates is $2U_o$. The flow is through a porous medium where the Darcy model is assumed which accounts for the drag exerted by the porous medium (Khaled, 2003). From the nature of the problem, $\frac{\partial(\cdot)}{\partial x} = \frac{\partial(\cdot)}{\partial z} = 0$, for all quantities except pressure gradient which is assumed constant.

The equations of motion are the continuity equation given in equation (4.14) and the Navier-Stokes equation in presence of Lorentz force and absence of body forces per unit mass of the fluid given in equation (4.15). The general velocity vector of the fluid is given in general as

$$v(y,t) = u(y,t)i + v_0 j \tag{4.55}$$

The Hall term will be ignored here in applying Ohm’s law as it has no marked effect for small and moderate values of the magnetic field. Similarly, the induced magnetic field will be neglected by assuming a very small magnetic Reynolds number. Because of the conservation of mass, i.e. $\nabla \cdot V = 0$ and the presence of uniform suction and velocity component $v(y, t)$ is assumed to have a constant value v_o. From the equation of continuity, it is observed that, u', v' and w' are the velocity components in x, y and z- directions respectively. Then this implies $v' = w' = 0$ and $u' \neq 0$, then the continuity equation is satisfied. With this in mind, u' is independent of x' and this makes $[(V \cdot \nabla)V]$ in the Navier-stokes equation to vanish.

In order to derive the governing equations of the problem, the following assumptions are made-:

1. The fluid is finitely conducting,
2. the fluid flow is incompressible,
3. the fluid flow is unsteady hence the flow variables depend on time,
4. the component of the magnetic force in the direction of x-axis,
5. the magnetic field is along y-direction,
6. the fluid is electrically neutral i.e. there is no surplus electrical charge distribution present in the fluid,
7. the only body forces present are Lorentz forces,
8. the fluid flow is unidirectional in x - axis ,
9. the flow is laminar,
10. the viscous dissipation and joule heat are neglected,
11. the Hall effect and polarization effect are negligible.

Under the above assumptions the fluid motion is governed by momentum equation

$$\rho \frac{\partial u'}{\partial t'} = -\frac{dP'}{dx'} + \mu \frac{\partial^2 u'}{\partial y'^2} - \frac{\mu}{K} u' - \sigma B_o^2 u' \tag{4.56}$$

while temperature distribution is governed by the energy equation

$$\rho C_p \left(\frac{\partial T'}{\partial t'} + v_o \frac{\partial T'}{\partial y'} \right) = k \frac{\partial^2 T'}{\partial y'^2} \tag{4.57}$$

where K is the Darcy permeability and T' is the dimensionless temperature. On introduction of angle of inclination α equation (4.56) reduces to

$$\rho \frac{\partial u'}{\partial t'} = -\frac{dP'}{dx'} + \mu \frac{\partial^2 u'}{\partial y'^2} - \frac{\mu}{K} u' - \sigma B_o^2 u' \sin^2(\alpha) \tag{4.58}$$

where α is the angle between **V** and **B**. This means that, both fields can be assessed at any angle α for $0 \leq \alpha \leq \pi$.

The origin is taken at the centre of the channel while the coordinate x' and y'-

axes are parallel and perpendicular to the channel walls respectively. The fluid motion starts at rest at $t = 0$. With the no-slip condition at the plates means that $u = 0$ at $y = -h$ and $u = U_O$ at $y = h$. Also assume that the initial temperature of the fluid is T_1. Hence, the initial and boundary conditions of temperature are

$T = T_1$ at $t = 0$, $T = T_1$ at $y = -h, t > 0$ and $T = T_2$ *at* $y = h, t > 0$ (4.59)

The momentum equation (4.58) now reduces to the form

$$\rho\left(\frac{\partial u'}{\partial t'} + v_0\frac{\partial u'}{\partial y'}\right) = -\frac{dP'}{dx'} + \mu\frac{\partial^2 u'}{\partial y'^2} - \frac{\mu}{K}u' - \sigma B_o^2 u' \sin^2(\alpha) \quad (4.60)$$

Let l be the characteristic length. To solve equations (4.60) and (4.57) subject to the above named boundary conditions by introducing the following dimensionless variables and parameters-:

$$x' = xl,\ y' = yl,\ u' = uU_o,\ P' = P\rho U_o^2,\ t' = \frac{tl}{U_o},\ T = \frac{T' - T}{T_2 - T_1} \quad (4.61)$$

$$Ha = \frac{\sigma B_o^2 l^2}{\mu},\ \Pr = \frac{\mu C_p}{k},\ \mathrm{Re} = \frac{\rho U_o l}{\mu},\ S = \frac{v_o}{U_o},\ M = \frac{l^2}{K} \quad (4.62)$$

In terms of these dimensionless quantities, equations (4.60) and (4.57) may be written, after dropping all primes for convenience as

$$\frac{\partial u}{\partial t} + S\frac{\partial u}{\partial y} = -\frac{dP}{dx} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2} - \frac{M}{\mathrm{Re}}u - \frac{1}{\mathrm{Re}}G^2 u \quad (4.63)$$

$$\frac{\partial T}{\partial t} + S\frac{\partial T}{\partial y} = \frac{1}{\Pr \mathrm{Re}}\frac{\partial^2 T}{\partial y^2} \quad (4.64)$$

of equations (4.63) and (4.64) are assumed to be of the form $u(y, t)$ and $T(y, t)$. The initial and boundary conditions for the velocity are respectively:

$$t \le 0 : u = 0, t > 0 : u = 0, y = -1, u = 1, y = 1 \quad (4.65)$$

and the initial and boundary conditions for the temperature are given by

$$t \le 0 : T = 0, t > 0 : T = 1, y = +1, T = 0, y = -1 \quad (4.66)$$

The linear differential equations (4.63) and (4.64) are solved numerically using finite difference approach under the initial and boundary conditions (4.65) and (4.66) so as to determine the velocity and temperature distributions for different values of the parameters M , S and α. In this technique, derivatives occurring in the generated differential equations have to be replaced by their finite difference approximations. The Crank-Nicolson implicit method is used at two successive time levels. Here, the finite difference equations relating to the variables are obtained by writing the equations at the midpoint of the computational cell and subsequently replacing the different terms by their second order central difference approximation in the y- direction. On the other hand, the diffusion terms are replaced by the average of the central differences at two successive time-levels. The resulting block tri-diagonal system is solved using Thomas-algorithm. The computational domain is $0 < t <$ and $-1 < y < 1$ which is divided into intervals with step sizes $\Delta t = 0.0001$ and $\Delta y = 0.005$ for time and space, respectively. When smaller step sizes are used, they do not show significant change in the results and convergence of the scheme is assumed when every one of u, T, and their gradients for the last two approximations differ from unity by less than 10^6 for all values of y in $-1 < y < 1$ at every time step. All calculations have been carried out for $\frac{dP}{dx} = -5$, $Pr = 1$ and $Re = 1$ so that we can determine the velocity and temperature distribution for different values of M, S, G^* and α. Equations (4.63) and (4.64) in finite difference form are expressed respectively as

$$(1.0\times10^{-4})u_{i+1}^{j+1} + (2.5\times10^{-4})u_{i}^{j+1} - (1.0\times10^{-4})u_{i-1}^{j+1} = (-5.0\times10^{-7}S + 1.0\times10^{-4})u_{i+1}^{j}$$
$$-(1.5\times10^{-4} + 5.0\times10^{-9}M + 5.0\times10^{-9}G^2)u_{i}^{j} + (5.0\times10^{-7}S + 1.0\times10^{-4})u_{i-1}^{j} + 2.5\times10^{-8}$$

and

$$(1.0\times10^{-4})T_{i+1}^{j+1} + (2.5\times10^{-4})T_{i}^{j+1} - (1.0\times10^{-4})T_{i-1}^{j+1} = (-5.0\times10^{-7}S + 1.0\times10^{-4})Tu_{i+1}^{j}$$
$$-(1.5\times10^{-4})T_{i}^{j} + (5.0\times10^{-7}S + 1.0\times10^{-4})T_{i-1}^{j}$$

The numerical computation generated linearized system of equations based on our step sizes and results of these are achieved with the aid of MATLAB application software. Eight approximations have been used in this study and have satisfied the convergence criteria for all ranges of the parameters used here. The velocity distribution for various values of M, S, Hartmann numbers $G^* = 1$, $G^* = 4$ and $G^* = 6$ and angles of inclinations $\alpha = 30^o$ and $\alpha = 60^o$ are presented on the tables 4.1 to 4.5.

Table 4.1 Couette velocity distribution for $M = 1$ and $S = 1$ for various Hartmann number

Ha	α	J	$u(y) \times 10^{-4}$								
			-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
1	30^o	2	1.2400	1.2400	1.2400	1.2400	1.2400	1.2400	1.2400	1.2400	1.2400
		6	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499
	60^o	2	1.2400	1.2400	1.2400	1.2400	1.2400	1.2400	1.2400	1.2400	1.2400
		6	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499
4	30^o	2	1.2396	1.2399	1.2399	1.2399	1.2399	1.2399	1.2399	1.2399	1.2399
		6	1.2496	1.2498	1.2498	1.2498	1.2498	1.2498	1.2498	1.2498	1.2498
	60^o	2	1.2396	1.2396	1.2396	1.2396	1.2396	1.2396	1.2396	1.2396	1.2396
		6	1.2496	1.2496	1.2496	1.2496	1.2496	1.2496	1.2496	1.2496	1.2496
6	30^o	2	1.2397	1.2397	1.2397	1.2397	1.2397	1.2397	1.2397	1.2397	1.2397
		6	1.2497	1.2497	1.2497	1.2497	1.2497	1.2497	1.2497	1.2497	1.2497
	60^o	2	1.2392	1.2392	1.2392	1.2392	1.2392	1.2392	1.2392	1.2392	1.2392
		6	1.2491	1.2491	1.2491	1.2491	1.2491	1.2491	1.2491	1.2491	1.2491

Table 4.2 Couette velocity distribution for $M = 2$ and $S = 0$ for various Hartmann number

Ha	α	J	$u(y)$								
			-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
1	30^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086
	60^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086
4	30^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085
	60^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4084	0.4084	0.4084	0.4084	0.4084	0.4084	0.4084	0.4084	0.4084
6	30^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085
	60^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086

Table 4.3. Couette velocity distribution for $M = 0$ and $S = 0$ for various for various Hartmann number

Ha	α	J	$u(y)$								
			-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
1	30^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086
	60^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086	0.4086
4	30^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085
	60^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4084	0.4084	0.4084	0.4084	0.4084	0.4084	0.4084	0.4084	0.4084
6	30^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085	0.4085
	60^o	2	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
		6	0.4083	0.4083	0.4083	0.4083	0.4083	0.4083	0.4083	0.4083	0.4083

Table 4.4 Couette velocity distribution for $M = 2$ and $S = 1$ for various Hartmann number

Ha	α	J	$u(y) \times 10^{-4}$								
			-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
1	30^o	2	1.2000	1.2000	1.2000	1.2000	1.2000	1.2000	1.2000	1.2000	1.2000
		4	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495
		6	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499
	60^o	2	1.2399	1.2399	1.2399	1.2399	1.2399	1.2399	1.2399	1.2399	1.2399
		4	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495
		6	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499	1.2499
4	30^o	2	1.2398	1.2398	1.2398	1.2398	1.2398	1.2398	1.2398	1.2398	1.2398
		4	1.2494	1.2494	1.2494	1.2494	1.2494	1.2494	1.2494	1.2494	1.2494
		6	1.2498	1.2498	1.2498	1.2498	1.2498	1.2498	1.2498	1.2498	1.2498
	60^o	2	1.2396	1.2396	1.2396	1.2396	1.2396	1.2396	1.2396	1.2396	1.2396
		4	1.2492	1.2492	1.2492	1.2492	1.2492	1.2492	1.2492	1.2492	1.2492
		6	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495	1.2495
6	30^o	2	1.2397	1.2397	1.2397	1.2397	1.2397	1.2397	1.2397	1.2397	1.2397
		4	1.2493	1.2493	1.2493	1.2493	1.2493	1.2493	1.2493	1.2493	1.2493
		6	1.2496	1.2496	1.2496	1.2496	1.2496	1.2496	1.2496	1.2496	1.2496
		2	1.2389	1.2389	1.2389	1.2389	1.2389	1.2389	1.2389	1.2389	1.2389
		4	1.2484	1.2484	1.2484	1.2484	1.2484	1.2484	1.2484	1.2484	1.2484

Table 4.5 Couette velocity distribution for $M = 5$ and $S = 2$ for various Hartmann number

Ha	α	J	$u(y) \times 10^{-4}$								
			-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
1	30^o	2	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019
		4	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019
		6	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019
	60^o	2	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019
		4	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019
		6	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019
4	30^o	2	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019
		4	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019
		6	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019	1.2019
	60^o	2	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017
		4	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017
		6	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017
6	30^o	2	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017
		4	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017
		6	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017	1.2017
		2	1.2014	1.2014	1.2014	1.2014	1.2014	1.2014	1.2014	1.2014	1.2014
		4	1.2014	1.2014	1.2014	1.2014	1.2014	1.2014	1.2014	1.2014	1.2014

Tables 4.1 to 4.5 shows effect of porosity parameter M and suction/injection parameter S on the time development of velocity u with increase in Hartmann number G^*. It is evident that, increase in Hartmann number decreases the velocity u and its steady state time with increase in the angles of inclination. The velocity component u reaches the steady state monotonically with time in table 4.1 when are evaluated at $M = 1$ and $S = 1$. When $S = 0$ (suction suppressed), increasing the porosity parameter M has no marked effect on velocity component u as shown on table 4.2 and 4.3.

From table 4.1 and table 4.4, increase in porosity parameter M results into decrease in velocity u with higher values of Hartmann number though this velocity reaches the steady state monotonically with time. In table 4.5, increasing the suction decreases the velocity u and this reaches the steady state monotonically with time and this is due to the convection of the fluid from regions in the lower half to the centre of the channel which has high fluid speed.

The temperature distribution for $S = 1$ and $S = 2$ for various time t are presented on the tables 4.6 and 4.7.

Table 4.6. Couette temperature distribution for $S = 1$ at different values of y for various time t

	$T(y)$								
j	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
0	8.1654e-167	1.2898e-148	2.0375e-130	3.2184e-112	5.0839e-940	8.0307e-76	1.2687e-57	2.0038e-39	3.1653e-21
2	1.1256e-161	1.4034e-143	1.6949e-125	1.9635e-107	2.1485e-89	2.1639e-71	1.9106e-53	1.3245e-35	5.0345e-18
4	2.5780e-157	2.5361e-139	2.3408e-121	1.9875e-103	1.5051e-85	9.6518e-68	4.7532e-50	1.4399 e-32	1.3015e-15
6	2.3548e-153	1.8270e-135	1.2881e-117	8.01106e-100	4.1947e-82	1.7107e-64	4.6892e-47	6.1841e-30	1.3173e-13

Table 4.7. Couette temperature distribution for $S = 2$ at different values of y for various time t

	$T(y)$								
j	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
0	-2.1290e-	-2.3408e-99	-2.5738e-87	-2.8299e-75	-3.1115e-63	-3.4211e-51	-3.7616e-39	-4.1359e-27	-4.5475e-15
2	-5.9277e-	-5.1474e-95	-4.3307e-83	-3.4957e-71	-2.6663e-59	-1.8733e-47	-1.1556e-35	-5.6185e-24	-1.5223e-12
4	-2.7108e-	-1.8557e-91	-1.1919e-79	-7.0416e-68	-3.7100e-56	-1.6550e-44	-5.6683e-33	-1.1941e-21	-7.5451e-11
6	-4.8870e-	-2.6327e-88	-1.2880e-76	-5.5519e-65	-2.0120e-53	-5.6636e-42	-1.0661e-30	-9.5493e-20	1.3486e-9

Tables 4.6 and 4.7 shows the effect of suction/injection parameter S on the time t development of temperature T. Increasing S decreases the temperature at the centre of the channel for all values of t. This fact is true and its due to the influence of convection in the pumping of the fluid from the cold lower half towards the centre of the channel.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

In this thesis steady and unsteady laminar viscous MHD fluid flow under the influence of transverse magnetic field is studied for both Poisseuille flow and Couette flow and the resultant partial differential equations solved analytically or numerically. Laminar steady Couette viscous incompressible fluid between two infinite parallel plates when the upper plate is moving with constant velocity and the lower plate is held stationary under the influence of transverse magnetic field is discussed. The resulting governing partial differential equations were solved analytically by Sumudu Transform for the linear differential equation with constant coefficients. The analysis of this showed that, the velocity profile decreases as the Hartmann number and magnetic inclination increases. This approach can be used to obtain solutions of ordinary differential equations in Astronomy, Physics and in controlling engineering problems.

The motion of two dimensional steady Poiseuille laminar flow of a viscous MHD incompressible fluid between two infinite parallel porous plates under the influence of uniform transverse magnetic field and with constant pressure gradient was examined. Both the lower plate and the upper plates were assumed porous where the fluid entered the flow region through the lower plate and left through the upper plate with constant velocity. The resulting coupled differential equations were solved numerically by using finite difference approach. The resulting block tri-diagonal system was solved using Thomas-algorithm and the velocity profiles obtained expressed in terms of Hartmann number. The results showed that, velocity of the fluid changes with varied Hartmann numbers. An increase in the Hartmann number leads to a decrease in velocity distribution. This is due to Lorentz force generated by the application of constant transverse magnetic field which offers resistance opposing the fluid motion and hence decreasing the flow.

Unsteady MHD Couette laminar flow of viscous incompressible fluid between two infinite parallel porous plates in presence of uniform magnetic field was studied. The upper and lower plates were maintained at two different but constant
which has high fluid velocity. It was also shown that, increasing the suction parameter *S* decreased the temperature distribution of the channel at the centre for all values of *t* and this was due to convection influence in the pumping action of the fluid from the lower half of the channel to the centre.

The solution of the resulting partial differential equations can act as a good approximation of some practical situations like flow meters, heat exchangers and pipes that connect such system components. The cooling of such devices are achieved by having the porous surface through which a coolant is forced. The coolant can either be a liquid or a gas. The solutions obtained in this study can therefore be of paramount importance for the design of the wall and cooling arrangements of such devices.

5.2. Recommendations

Effects of various parameters on steady and unsteady Magnetohydrodynamic fluid flow for both Poiseuille and Couette fluid flow under the influence of uniform magnetic field has been studied. The study of variable magnetic field on porous and non-porous infinite parallel plates or channels in which Hartmann number in combination with other parameters may be important for further development and analysis. These may be experimental or theoretical in approach. Such specific areas of study are:-

1. Flow involving variable transverse magnetic field applied at various angles of inclinations for Poiseuille fluid flow on porous channels.
2. Flow involving variable inclined magnetic field applied to Couette fluid flow when both plates are porous with variable suction or injection.
3. Steady and unsteady MHD flow with variable viscosity and thermal conductivity and taking into account viscous dissipation, Hall effect and polarization effects on the fluid when variable magnetic field are involved.
4. Steady MHD incompressible fluid flow in three dimensions with variable magnetic field problems.

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