DECLARATION

Declaration by the Candidate

This thesis is my original work and has not been presented for a degree in any other university. No part of this thesis may be reproduced without the prior written permission of the author and/or University of Eldoret.

Kenneth Kipchumba Sirma

Signature …..……………………

Date: ……..…………………

Approval by Supervisors

This thesis has been submitted for examination with our approval as University Supervisors.

Prof Kapil M. Khanna

Signature …………………… Date: ……………………..

Physics department

University of Eldoret, Kenya

Dr. Samuel L. Chelimo

Signature …………………………. Date: ………………..……

Physics department

University of Eldoret, Kenya.

DEDICATION

To my beloved mum Grace for her unconditional love, advice and support.

To our son Gael you are a blessing. My wife Jacinta, for her love and care.

ABSTRACT

Properties of nuclear matter and large finite nuclei have been studied using different types of nucleon-nucleon (NN) interaction potential. In this study, focus is on the thermodynamic properties of a large nucleus of Rhenium ($^{186}_{75}Re$) using NN-interaction potential. Rhenium has been chosen because of its applications in technology, it's used in engines of airplanes, missiles and high temperature thermo-couplers thus it may equally have an application at lower temperatures. The NN-interaction is used as a perturbation and general methods of quantum mechanics as well as many-body techniques have been used to calculate the energy, the heat capacity and the entropy. It is found that the heat capacity exhibits a phase transition at a critical temperature of 0.144K. The heat capacity increases linearly with increase in excitation energy at constant temperature and this is due to changes in the internal energy of nuclei. The entropy of the nuclei increases linearly with temperature and becomes zero at absolute 0K. These results suggests the possibility of Rhenium, a fermi-system exhibiting superfluid properties below the critical temperature T_c~0.144K.

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LIST OF SYMBOLS

Symbol	Description
a^{+}	Creation operator
a	Annihilation
E	Energy
H	Hamiltonian
H_o	Unperturbed Hamiltonian
$H^{'}$	Perturbation
h	Planck’s Constant ($6.626\text{x}10^{-34}$JS)
k_B	Boltzmann’s Constant ($1.3807\text{x}10^{-23}$J/K)
M	Mass of nucleus
R	Radius of the Nucleus
T	Kinetic energy operator
V	Potential energy Operator
x	Displacement operator
β, γ	Perturbation parameters

ω	Frequency
m_α	Mass of relevant meson
η	Density
ϕ_o	Unperturbed wave function
φ	Perturbed wave function
J	Total angular momentum
L	Orbital Angular momenta
M_1, M_2	Majorama space exchange force
P_x	Majorama space exchange operator
s	Entropy
S	Total spin
S_{12}	Tensor force term between two particles
W	Wigner force
$\mathcal{T}_i$	Isospin
σ_i	Spin

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ACRONYMS

NN	Nucleon-Nucleon
CPEP	Contemporary Physics Education Project
Eq	Equation
$^{186}_{75}Re$	Rhenium (having 111 Neutrons and 75 Protons)

CHAPTER ONE

INTRODUCTION

1.1 Background information

The present concept of the atom, gained after innumerable scientific investigations over many decades is that it is composite, hence divisible and synthesizable. It is now universally accepted that the atom consists of two distinct regions. The tiny central core, called the nucleus having a radius of the order of a few fermis (~10^{-15} m), and the relatively extensive sorounding space reffered to as the outer sphere which has a radius of the order of 10^{-10} M. Every atom in turn is made of three different particles the electrons, protons and neutrons. The single exception to this is the atom of ordinary hydrogen which contains only an electron and a proton but no neutron. It is further established that while electrons occupy the outer sphere, the protons and neutrons are inside the nucleus.

1.2 General Survey

Discovery of neutron by Chadwick completed more or less on the knowledge of structure of the nucleus. After this discovery the nucleus was proposed to be composed of neutrons and protons, which have almost the same mass. Protons and neutrons commonly referred to as nucleons are spin-half particles, which obey Fermi-Dirac statistics distribution (Roy and Nigam 1967).

The fact that nucleons in a nucleus are confined to a small region of space, is therefore, convincing proof of the presence of very strong attractive and nuclear forces binding them together. There are several kinds of interactions that are presently known and existing between different bodies and particles (Soita, 2007)

i. Gravitational interaction between different celestial bodies
ii. Weak interactions, which give rise to the emission of β-particles.
iii. Electromagnetic interaction, which exists between charged particles by virtue of their charge, current and magnetic moments.
iv. Strong interaction, which binds the nucleons in a nucleus.

In the past, there has been a tremendous experimental effort devoted to the study of scattering of protons on protons, and neutrons on protons. Since the neutron as a target is not available, the neutron-neutron scattering was inferred mostly from the scattering of protons on deuterons. All this effort leads to a large database of cross-sections and phase shifts that provide the most extensive information on the binary interactions between nucleons. There have also been numerous attempts to model the interaction between nucleons by different kinds of potentials (Hagen *et al.*, 2007). A successful way of describing nuclear interactions is to construct one potential for the whole nucleus instead of considering all its nucleon components that may result to a complex matrix to solve.

There has been growing interest towards the calculation of the properties of nuclear matter and large finite nuclei using different types of NN-interaction potentials. (Moszkowski. 1970; Lassey 1972; Khanna and Barhai 1975; Krewald *et al*., 1976; and Dean *et al*., 2003). Particularly, the ground state energy of the nuclear matter has been calculated using various types of nucleon-nucleon potentials.

A velocity dependent effective potential of *s*-wave interaction was proposed (Dzhibuti *et al*., 1969) to calculate the properties of nuclear matter, especially binding energy and radii of different nuclei from ^{4}He to ^{208}Pb.

A set of NN interaction (Dean *et al*., 2003) is characterized by the existence of a strongly repulsive core at short distances, with a characteristic radius $\simeq$ 0.5 ƒm. The interaction obeys several fundamental symmetries, such as translational, rotational, spatial reflection, time-reversal invariance, and exchange symmetry. It also has a strong dependence on quantum numbers such as total spin S and isospin T, through the nuclear tensor force that arises, for instance, from one-pion exchange. It also depends on the angles between the nucleon (pairs) spins and separation vector. The tensor force thus mixes different angular momenta L of the two-body system, that is, it couples two-body states with total angular momentum J=L–1 and J=L+1. For instance, for a proton-neutron two-body state, the tensor force couples the states $^3S_1$ and $^3D_1$, where the standard spectroscopic notation $^{2S+1}L_J$ has been used.

There is no unique prescription as how to construct a NN-interaction, a description of the interaction in terms of various meson exchanges is at present the most quantitative representation (Wiringa *et al*., 1995; and Machleidt 2001) in the energy regime of nuclear structure physics. It should be emphasized that meson exchange is an appropriate picture at low and intermediate energies.

1.3 Problem Statement

Several interaction potentials have been used to study the nucleon properties of matter that may include velocity-dependent, spin-dependent and density-dependent potential, but in this study second quantization approach has been used in studying the nuclear properties of matter.

Considering an interaction between nucleons via a potential $V = \beta\mathcal{X}^3 + \gamma\mathcal{X}^4$ and using it in a many body quantum mechanical system Hamiltonian $H = H_o + V$, where H_o is the unperturbed Hamiltonian and V is the interaction between the particles, the expression for the energy eigenvalues will be derived.

Using Bogoliubov transformation which provides a rather powerful tool for nuclear many-body problem, the Hamiltonian will be diagonalized to get the energy eigenvalues. Hence we get the energy of the system that will be used to investigate its thermodynamic properties.

1.4 General Objective

To study thermodynamic properties of $^{186}_{75}\mathrm{Re}$ using NN-interaction potential.

1.4.1 Specific Objectives

1. To derive an expression for the Energy E and the Binding Energy per Nucleon $\left(\frac{\mathrm{E}}{\mathrm{N}}\right)$ for $^{186}_{75}\mathrm{Re}$.
2. To determine expression of the Heat Capacity C and Entropy S using the energy of $^{186}_{75}\mathrm{Re}$.
3. To determine the Critical Transition Temperature T_c of $^{186}_{75}\mathrm{Re}$.

1.5 Justification

Recently, there has been an increase on the research, both theoretical and experimental, to study the properties of matter to explore potential areas of applications in technology.

However, there have been also different theoretical approaches on this study but due to the dynamics of nuclear properties and their applications, new approaches may yield more precise results that impacts on further developments on its applications in technology.

$^{186}_{75}Re$ is an important element because of its applications in technology, its used in high-temperature thermocouplers, refractory metal components of missiles, igniters of flash bulbs and oven filaments, thus the study of this material will provide more precise information that will help improve its applications.

CHAPTER TWO

THEORY AND LITERATURE REVIEW

2.1 Background information

One of the major challenges in nuclear theory is to understand and predict the structure of nucleonic matter based on microscopic NN and many nucleon interactions. In recent years, there has been significant progress in exact calculations of ground and excited states of light nuclei based on various high-precision interactions fitted to NN data (Kamada *et al*., 2001; Pieper and Wiringa 2001; Wiringa *et al*., 2002; Nogga *et al*., 2002; Navratil *et al*., 2000; Navratil and Ormand 2003; Nogga *et al*., 2006). These results clearly show that three-nucleon forces (3NFs) contribute significantly: Without 3NFs, the binding energies depend strongly on the NN potential used, which can be traced to scheme and model dependences in any theory restricted to NN interactions. The study of 3NFs in systems beyond the lightest nuclei is an important goal. This requires a flexible technique to solve the many-body problem including NN and 3N interactions.

A lot of interest has been devoted in the calculation of the properties of nuclear matter and large finite nuclei using different types of NN-interactions. (Moszkowski, 1970; Lassey 1972; Khanna and Barhai 1975; Krewald *et al*., 1976; Dean *et al*., 2003). Particularly, the ground state energy of the nuclear matter has been calculated using the various types of NN potentials.

2.2 The Velocity-Dependent Interaction Potential.

A velocity dependent effective potential of *s*-wave interaction with one free parameter was proposed (Dzhibuti and MamasaRhlisov, 1969) to calculate the properties of nuclear matter, especially binding energy and radii of different nuclei from ^{4}He to ^{208}Pb. The velocity-dependent effective *s*-wave interaction is defined by,

$$V_{eff}(r) = \tfrac{1}{2}\left\{ V_{eff}(r) e^{\left[-a\frac{\partial}{\partial r}\right]} + e^{\left[a\frac{\partial}{\partial r}\right]} V_{eff}(r) \right\}_{a \to r} - \lambda(A)\frac{\hbar^2}{M}\left\{\partial(r)\nabla^2 + \nabla^2\partial(r)\right\} \qquad (2.0)$$

$$V_{real}(r) = -V_0 \frac{e^{-ur}}{ur} \qquad (2.1)$$

where $V_{real}(r)$ is the initial realistic potential parametrized in accordance with the two nucleon problem in vacuum and $a \to r$ is the substitution that must be made in the two-particle matrix elements after acting with the operator ($e^{-a\frac{\partial}{\partial r}}$) on the wave function of the pair from the right, and with the operator ($e^{a\frac{\partial}{\partial r}}$) on the analogous function from the left. The second term on the right–hand-side of Eqn (2.0) contains only the additional parameter λ, and this represents phenomenological the multiparticle effects or many body interactions. In the first version of such a potential $V_{real}(r)$ can be taken in the Yukawa form (Krewald *et al.*, 1976) the effective interaction in this case will be referred to as VY with identical parameters for singlet and triplet central forces. With parameters from the free nucleon-nucleon scattering at low energies, written as,

$$V_0 = 48.1 \text{ MeV}; \qquad u = 0.86 \text{ F}^{-1} \qquad (2.2)$$

F = 1 *fm* = 10^{-15} m.

In the second version of this potential, $V_{real}(r)$ can be taken in the Gaussian form and the effective interaction in this case will be referred to as VG, i.e.

$$V_{real}(r) = [\alpha(\mathcal{T}_1.\mathcal{T}_2) + \alpha_{\sigma\mathcal{T}}(\sigma_1.\sigma_2)(\mathcal{T}_1.\mathcal{T}_2)]\, e^{-\frac{r^2}{r_o^2}} \qquad (2.3)$$

where σ_i and $\mathcal{T}_i$ are the spin and isospin parameters

$\alpha_{\mathcal{T}}$= 2.096 MeV; $a_{\sigma\,\mathcal{T}}$= 7.767 MeV; r_o= 2.18 F (2.4)

In the case of VY interaction, λ turns out (Hassan *et al.*, 1978) to be, λ = 1.3 F^3, and for VG interaction, λ = 3.9 F^3.

2.3 Spin-Dependent Interaction Potential

A set of NN interaction (Dean *et al*., 2003) is characterized by the existence of a strongly repulsive core at short distances, with a characteristic radius $\simeq$ 0.5 *fm*. The interaction obeys several fundamental symmetries, such as translational, rotational, spatial reflection, time-reversal invariance, and exchange symmetry. It also has a strong dependence on quantum numbers such as total spin *S* and isospin *T*, through the nuclear tensor force that arises, for instance, from one-pion exchange. It also depends on the angles between the nucleon (pairs) spins and separation vector. The tensor force thus mixes different angular momenta *L* of the two-body system, that is, it couples two-body states with total angular momentum *J=L–1* and *J=L+1*. For instance, for a proton-neutron two-body state, the tensor force couples the states $^3S_1$ and $^3D_1$, where the standard spectroscopic notation $^{2S+1}L_J$ has been used.

There is no unique prescription as how to construct a NN-interaction, a description of the interaction in terms of various meson exchanges is at present the most quantitative representation (Krewald *et al*., 1976; Lassey *et al*., 1972) in the energy regime of nuclear structure physics. It should be emphasized that meson exchange is an appropriate picture at low and intermediate energies. Thus in the NN-interaction, it may be enough to include central, spin-spin, tensor and spin-orbit interaction terms. Hence the interaction (Dean *et al*., 2003) omitting isospin, can be written as,

$$V_{(r)} = \left\{ C_c^0 + C_c^1 + C_\sigma \sigma_1.\sigma_2 + C_T \left[1 + \frac{3}{m_\alpha r} + \frac{3}{(m_\alpha r)^2} \right] S_{12}(r) + C_{SL} \left[\frac{3}{m_\alpha r} + \frac{3}{(m_\alpha r)^2} \right] L.S \right\} \frac{e^{-m_\alpha r}}{m_\alpha r}$$

(2.5)

Where $m_\propto$ is the mass of the relevant meson and S_{12} is the tensor force term,

$$S_{12(r)} = \sigma_1.\sigma_2 r^2 - (\sigma_1.r)(\sigma_2.r) \quad (2.6)$$

Where σ is the standard operator for spin $^1/_2$ particles within meson-exchange models, we may have the exchange of $\pi, \rho, \sigma, \omega, \delta$ and ŋ mesons. As an example, the co-efficient for the exchange of a π meson are $C_\sigma = C_T = \left(\frac{g_{NN\pi}^2}{4\pi}\right)\left(\frac{m_\pi^3}{12 m_N^2}\right)$, and $C_C^0 = C_C^1 = C_{SL} = 0$ with the experimental (Machleidt 2001) value for $g_{NN\pi}^2 \simeq$ 13-14, and M_N is the mass of a nucleon and it will be taken as the average of the proton and neutron masses.

2.4 Density-Dependent Interaction Potential

There is another nucleon-nucleon interaction (Khanna *et al*., 1973) that can be used to study the infinite nuclear matter. It is of the form,

$$V = V_A + V_B \quad (2.7)$$

where $V_A = -V_1(W_1 + M_1 P_x) e^{\left[-(\frac{r_{12}}{\mu_1})^2\right]} + V_2(W_2 + M_2 P_x) e^{\left[-(\frac{r_{12}}{\mu_2})^2\right]}$ (2.8)

$$V_B = G_S(4\eta)\delta(r_{12}) \quad (2.9)$$

Here $G_S(4\eta) \simeq C\kappa_F^2(r)$ is the local Fermi momentum, *r* is the distance from the centre of a very large hypothetical finite nucleus having the density of a nuclear matter whose surface thickness is zero (Elton 1961) *C* is a constant that can be determined from the nuclear matter saturation conditions and η is the density of one kind of nucleon.

The V_A part of the interaction represents the Volkov potential (Volkov 1965; Zofka *et al.*, 1970) for

$$W_1 = W_2 = 0.4 \text{ and } W_1 + M_1 = W_2 + M_2 = 1 \quad (2.10)$$

The V_B part represents the density-dependent delta interaction and the value of $K_F^2 = (4\eta)^{2/3}$ Now two sets of calculations can be done by having two types of potentials. The first can be in which the exchange parameters M_1 and M_2 are considered equal while in the second set the original Volkov potential (Volkov, 1965) with unequal exchange mixtures can be used. Here W represents Wigner force (which is a short-range repulsive) and M represents Majorama space exchange force and P_x is the Majorama space exchange operator which gives +1 for states with even ℓ and -1 for states with odd ℓ.

Thus for the first set of calculations, it can be written,

$$W_1 = W_2 = 0.5, \text{ and } W_1 + M_1 = W_2 + M_2 = 1 \quad (2.11)$$

In this case, V_A given in Eqn 2.8 becomes,

$$V_A = {}^1/_2 (1 + P_x)\ \mho\ (\gamma_{12}) \quad (2.12)$$

Where

$$\mho\ (\gamma_{12}) = -V_1 e^{\left[-(\frac{r_{12}}{\mu_1})^2\right]} + V_2\ e^{\left[-(\frac{r_{12}}{\mu_2})^2\right]} \quad (2.13)$$

For the second set the values given in Eqn (2.10) are used and hence gives

$$V_A = (0.4 + 0.6\ P_x)\ \mho\ (\gamma_{12}) \quad (2.14)$$

The parameters of the Volkov potential (Volkov, 1965; Zofka, 1970) are,

$$V_1 = 83.3 \text{ MeV; V2} = 144.9 \text{ MeV; } \mu_1 = 1.6\ fm,\ \mu_2 = 0.81 fm \quad (2.15)$$

There is another nucleon-nucleon interaction (Khanna 2008) of the form V = $\beta\chi^3 + \gamma\chi^4$ where V is the potential that is inserted into the Hamiltonian $H = H_o + V$ where H_o is the unperturbed Hamiltonian, and V is the interaction potential, β and γ are perturbation parameters and $\chi = \frac{1}{\alpha 2^{1/2}}(a + a^{+})$ where a, a^{+} is the annihilation and creation operator and $\alpha^{-1} = \left(\frac{\hbar}{m\omega}\right)^{1/2}$ this potential has been considered in this study.

2.5 Many Body Theory

Many-body theory is used to study Hamiltonians which have terms other than the kinetic energy term. In a given nucleus made up of more than two nucleons, the total nucleon interaction is the sum of the interaction between all pairs of nucleons. For a nucleus of *A* nucleons the Hamiltonian takes the form (Hagen *et al.*, 2010)

$$H = \sum_{i=1}^{A} T(i) + \sum_{i<j}^{A} V(ij) \tag{2.16}$$

Where *T* denotes the single-particle kinetic energy operator and *V* the two nucleon potential. The restriction *i*<*j* in the second sum takes care of the fact that the interaction has to be summed only counting each pair once. The Schrodinger equation

$$H\varphi(1,2,\dots,A) = E\varphi(1,2,\dots,A) \tag{2.17}$$

can not generally be solved in a straight forward manner.

2.6 Thermodynamic Properties of Nuclei

One of interesting problems in the study of small system phase transition is the possible existence of a phase transition from a hadronic phase to quark-gluon plasma in high energy physics as shown in Appendix B. The yellow part of Appendix B shows that the phase transition between the nuclear liquid and a gas of nucleons is not the only phase transition that heavy ion scientists are studying.

At even higher temperatures and densities, the nucleons themselves can undergo a phase transition, also existence of a phase transition from normal state to a superfluid state which is accompanied by rather drastic changes in both the thermodynamic equilibrium and thermal transport properties of a superfluid.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

$^{186}_{75}$Re is studied using the an interactive potential $H^1 = \beta\mathcal{X}^3 + \gamma\mathcal{X}^4$ which is added to the unperturbed Hamiltonian to create unharmonicity in the system, second quantization techniques are employed to solve the energy of the system.

3.2 Energy of the System

The Hamiltonian of the system is given as

$$H = H_0 + V \tag{3.1}$$

where H_0 = unperturbed Hamiltonian which will be the kinetic energy of the system. (Hagen *et al.,* 2010).

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}\, kx^2 \tag{3.2}$$

and

$$V = H^1 = \beta\mathcal{X}^3 + \gamma\mathcal{X}^4 \tag{3.3}$$

V is the perturbation potential due to interaction between the nucleons. Using the concept of many-body systems we can write

$$\langle n|H|n\rangle = \langle n|H_0|n\rangle + \langle n|H^1|n\rangle \tag{3.4}$$

The right hand side of Eq 3.4 can be expressed as follows

$$\langle n|H_0|n\rangle = E_0\langle n|n\rangle \tag{3.5}$$

where $E_0 = \left(\mathrm{n} + \frac{1}{2}\right)\hbar\omega$, n=0,1,2,3… (3.6)

$$\langle \mathrm{n}|H^1|\mathrm{n}\rangle = \langle \mathrm{n}|\beta\mathcal{X}^3 + \gamma\mathcal{X}^4|\mathrm{n}\rangle$$

$$= \langle \mathrm{n}|\beta\mathcal{X}^3|\mathrm{n}\rangle + \langle \mathrm{n}|\gamma\mathcal{X}^4|\mathrm{n}\rangle$$

$$= \beta\langle \mathrm{n}|\mathcal{X}^3|\mathrm{n}\rangle + \gamma\,\langle \mathrm{n}|\mathcal{X}^4|\mathrm{n}\rangle$$

Whereβ, γ are perturbation parameters?

The second quantization creation and annihilation operators are given as

$$a^+ = \frac{1}{\sqrt{2}}\left(\xi - \frac{\partial}{\partial\xi}\right) = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x - \frac{ip}{m\omega}\right) \quad (3.7)$$

$$a = \frac{1}{\sqrt{2}}\left(\xi + \frac{\partial}{\partial\xi}\right) = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x + \frac{ip}{m\omega}\right) \quad (3.8)$$

With the following properties

$$a^+|n\rangle = (n+1)^{\frac{1}{2}}|n+1\rangle$$
$$a|n\rangle = n^{\frac{1}{2}}|n-1\rangle \quad (3.9)$$

Using these operators, Eq. 3.7 and 3.8, the displacement operator become,

$$x = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}\left(a + a^+\right) = \left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}}\frac{1}{\sqrt{2}}\left(a + a^+\right) \quad (3.10)$$

Let $\alpha^{-1} = \left(\frac{\hbar}{\mathrm{m}\omega}\right)^{1/2}$ therefore $\mathcal{X} = \frac{1}{\alpha\sqrt{2}}(a + a^+)$;

$$\mathcal{X}^3 = \frac{1}{\alpha^3\sqrt{8}}(a + a^+)^3 \quad (3.11)$$

$$\mathcal{X}^4 = \frac{1}{4\alpha^2}(a + a^+)^4 \qquad (3.12)$$

$$\langle \mathrm{n}|\mathrm{H}^1|\mathrm{n}\rangle = \frac{\beta}{\alpha^3\sqrt{8}}\langle \mathrm{n}|(a + a^+)^3|\mathrm{n}\rangle + \frac{\gamma}{4\alpha^2}\langle \mathrm{n}|(a + a^+)^4|\mathrm{n}\rangle \qquad (3.13)$$

working out the first part of Eq. 3.13 on the right hand side. We need to expand $(a + a^+)^3$ then substitute the below Bogoulibov transformation

$$\begin{aligned} a &= u_R \ell_R + v_R \ell_R^+ \\ a^+ &= u_R \ell_R^+ + v_R \ell_R \end{aligned} \qquad (3.14)$$

ℓ_R , ℓ_R^+ are new operator's and a, a^+ are old operator's.

$$\varphi = a^+(u_R + v_R a^+ a^+)\left|0\right>$$

$$<\varphi\left|H\right|\varphi> = <0\left|a(u_R + v_R aa)H^1(u_R + v_R a^+ a^+)a^+\right|0>$$

$$u_R^2 + v_R^2 = 1 \qquad (3.15)$$

Values of u_R and v_R are

$$\begin{aligned} u_R &= 0, v_R = 1 \\ u_R &= 1, v_R = 0 \end{aligned} \qquad (3.16)$$

$$u_R = v_R = \frac{1}{\sqrt{2}} \qquad (3.17)$$

Expanding the expression $(a + a^+)^3$ we end up getting

$$\left(a + a^+\right)^3 = aaa + aaa^+ + aa^+a^+ + a^+aa + a^+aa + a^+aa^+ + a^+a^+a + a^+a^+a^+$$

(3.18)

Now substituting Eq. (3.14) into each of Eq. (3.18)

For

$$aaa=\left(u_R\ell_R+v_R\ell_R^+\right)\left(u_R\ell_R+v_R\ell_R^+\right)\left(u_R\ell_R+v_R\ell_R^+\right)$$

$$=\begin{bmatrix} u_R^3\ell_R\ell_R\ell_R+u_R^2v_R\ell_R\ell_R\ell_R^++u_R^2v_R\ell_R\ell_R^+\ell_R+v_R^2u_R\ell_R\ell_R^+\ell_R^+ \\ +u_R^2v_R\ell_R^+\ell_R\ell_R+u_Rv_R^2\ell_R^+\ell_R\ell_R^++v_R^2u_R\ell_R^+\ell_R^+\ell_R+v_R^3\ell_R^+\ell_R^+\ell_R^+ \end{bmatrix}$$

$$\frac{\beta}{\alpha^3\sqrt{8}}\langle n,0\left|\begin{matrix} u_R^3\ell_R\ell_R\ell_R+u_R^2v_R\ell_R\ell_R\ell_R^++u_R^2v_R\ell_R\ell_R^+\ell_R+v_R^2u_R\ell_R\ell_R^+\ell_R^+ \\ +u_R^2v_R\ell_R^+\ell_R\ell_R+u_Rv_R^2\ell_R^+\ell_R\ell_R^++v_R^2u_R\ell_R^+\ell_R^+\ell_R+v_R^3\ell_R^+\ell_R^+\ell_R^+ \end{matrix}\right|n,0\rangle \quad (3.19)$$

Eqn. 3.21 gives zero because it has both odd number of creation and annihilation operators.

For $$aaa^+=\left(u_R\ell_R+v_R\ell_R^+\right)\left(u_R\ell_R+v_R\ell_R^+\right)\left(u_R\ell_R^++v_R\ell_R\right)$$

$$=\begin{bmatrix} u_R^3\ell_R\ell_R\ell_R^++u_R^2v_R\ell_R\ell_R\ell_R+v_Ru_R^2\ell_R\ell_R^+\ell_R^++u_Rv_R^2\ell_R\ell_R^+\ell_R \\ +u_R^2v_R\ell_R^+\ell_R\ell_R^++u_Rv_R\ell_R^+\ell_R\ell_R+v_R^2u_R\ell_R^+\ell_R^+\ell_R^++v_R^3\ell_R^+\ell_R^+\ell_R \end{bmatrix}$$

$$\frac{\beta}{\alpha^3\sqrt{8}}\langle n,0\left|\begin{matrix} u_R^3\ell_R\ell_R\ell_R^++u_R^2v_R\ell_R\ell_R\ell_R+v_Ru_R^2\ell_R\ell_R^+\ell_R^++u_Rv_R^2\ell_R\ell_R^+\ell_R \\ +u_R^2v_R\ell_R^+\ell_R\ell_R^++u_Rv_R\ell_R^+\ell_R\ell_R+v_R^2u_R\ell_R^+\ell_R^+\ell_R^++v_R^3\ell_R^+\ell_R^+\ell_R \end{matrix}\right|n,0\rangle \quad (3.20)$$

Eq. (3.20) gives zero.

For

$$aa^+a=\left(u_R\ell_R+v_R\ell_R^+\right)\left(u_R\ell_R^++v_R\ell_R\right)\left(u_R\ell_R+v_R\ell_R^+\right)$$

$$=\frac{\beta}{\alpha^3\sqrt{8}}\langle n,0\left|\begin{matrix}u_R^3\ell_R\ell_R^+\ell_R+u_R^2v_R\ell_R\ell_R^+\ell_R^++u_R^2v_R\ell_R\ell_R\ell_R+u_Rv_R^2\ell_R\ell_R\ell_R^+\\+u_R^2v_R\ell_R^+\ell_R^+\ell_R+u_Rv_R^2\ell_R^+\ell_R^+\ell_R^++u_Rv_R^2\ell_R^+\ell_R\ell_R+v_R^3\ell_R^+\ell_R\ell_R^+\end{matrix}\right|n,0\rangle$$

(3.21)

Eqn. 3.21 gives zero

$$aa^+a^+=\left(u_R\ell_R+v_R\ell_R^+\right)\left(u_R\ell_R^++v_R\ell_R\right)\left(u_R\ell_R^++v_R\ell_R\right)$$

$$=\frac{\beta}{\alpha^3\sqrt{8}}\langle n,0\left|\begin{matrix}u_R^3\ell_R\ell_R^+\ell_R^++u_R^2v_R\ell_R\ell_R^+\ell_R+u_R^2v_R\ell_R\ell_R\ell_R^++u_Rv_R^2\ell_R\ell_R\ell_R\\+u_R^2v_R\ell_R^+\ell_R^+\ell_R^++u_Rv_R^2\ell_R^+\ell_R^+\ell_R+u_Rv_R^2\ell_R^+\ell_R\ell_R^++v_R^3\ell_R^+\ell_R\ell_R\end{matrix}\right|n,0\rangle$$

(3.22)

This gives zero

For $a^+aa, a^+aa^+, a^+a^+a, a^+a^+a^+$ will give a result of zero. Thus the first part of Eqn. 3.13 will be zero.

The second part of Eqn. 3.13 on the right hand side is

$$\frac{\gamma}{4\alpha^2}\langle n,0\left|\left(a+a^+\right)^4\right|n,0\rangle \qquad (3.23)$$

Expanding the part that contains power i.e $\left(a+a^+\right)^4$

$$\left(a+a^+\right)^4=\left|\begin{matrix}aaaa+aaaa^++aaa^+a+aaa^+a^++aa^+aa+aa^+aa^++aa^+a^+a+aa^+a^+a^++\\a^+aaa+a^+aaa^++a^+aa^+a+a^+aa^+a^++a^+a^+aa+a^+a^+aa^++a^+a^+a^+a+a^+a^+a^+a^+\end{matrix}\right|$$

(3.24)

Using Eqn. 3.14 and substitute to Eqn. 3.24

$$aaaa = \left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R + v_R \ell_R^+\right)$$

$$= \frac{\gamma}{4\alpha^2}\langle n,0\left|\begin{matrix} u_R^4 \ell_R \ell_R \ell_R \ell_R + u_R^3 v_R \ell_R \ell_R \ell_R \ell_R^+ + u_R^3 v_R \ell_R \ell_R \ell_R^+ \ell_R + u_R^2 v_R^2 \ell_R \ell_R \ell_R^+ \ell_R^+ \\ + u_R^3 v_R \ell_R \ell_R^+ \ell_R \ell_R + u_R^2 v_R^2 \ell_R \ell_R^+ \ell_R \ell_R^+ + u_R^2 v_R^2 \ell_R \ell_R^+ \ell_R^+ \ell_R + v_R^3 u_R \ell_R \ell_R^+ \ell_R^+ \ell_R^+ \\ + u_R^3 v_R \ell_R^+ \ell_R \ell_R \ell_R + u_R^2 v_R^2 \ell_R^+ \ell_R \ell_R \ell_R^+ + u_R^2 v_R^2 \ell_R^+ \ell_R \ell_R^+ \ell_R + v_R^3 u_R \ell_R^+ \ell_R \ell_R^+ \ell_R^+ \\ + u_R^2 v_R^2 \ell_R^+ \ell_R^+ \ell_R \ell_R + u_R v_R^3 \ell_R^+ \ell_R^+ \ell_R \ell_R^+ + u_R v_R^3 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R + v_R^4 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ \end{matrix}\right| n,0\rangle$$

The following gives zero.

$$\left.\begin{matrix} u_R^4 \langle n,0 \left| \ell_R \ell_R \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots \\ u_R^3 v_R \langle n,0 \left| \ell_R \ell_R \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots \\ u_R^3 v_R \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots \\ u_R^3 v_R \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots \\ v_R^3 u_R \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots \\ u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots \\ v_R^3 u_R \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots \\ u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots \\ u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots \\ v_R^4 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots \end{matrix}\right] = 0$$

The following does not give a zero

$$
\left.\begin{array}{l}
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots (n+1)(n+2) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots (n+1)^2 u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n+1) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n+1) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n^2 u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n-1) u_R^2 v_R^2
\end{array}\right]
$$

(3.25)

For

$$aaaa^+ = \left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R^+ + v_R \ell_R\right)$$

$$
= \frac{\gamma}{4\alpha^2} \langle n,0 \left| \begin{array}{l}
u_R^4 \ell_R \ell_R \ell_R \ell_R^+ + u_R^3 v_R \ell_R \ell_R \ell_R^+ \ell_R^+ + u_R^3 v_R \ell_R \ell_R^+ \ell_R \ell_R^+ + u_R^2 v_R^2 \ell_R \ell_R^+ \ell_R^+ \ell_R^+ \\
+ u_R^3 v_R \ell_R^+ \ell_R \ell_R \ell_R^+ + u_R^2 v_R^2 \ell_R^+ \ell_R \ell_R^+ \ell_R^+ + u_R^2 v_R^2 \ell_R^+ \ell_R^+ \ell_R \ell_R^+ + u_R v_R^3 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ \\
+ u_R^3 v_R \ell_R \ell_R \ell_R \ell_R + u_R^2 v_R^2 \ell_R \ell_R \ell_R^+ \ell_R + u_R^2 v_R^2 \ell_R \ell_R^+ \ell_R \ell_R + u_R v_R^3 \ell_R \ell_R^+ \ell_R^+ \ell_R \\
+ u_R^2 v_R^2 \ell_R^+ \ell_R \ell_R \ell_R + u_R v_R^3 \ell_R^+ \ell_R \ell_R^+ \ell_R + u_R v_R^3 \ell_R^+ \ell_R^+ \ell_R \ell_R + v_R^4 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+
\end{array} \right| n,0 \rangle
$$

The following give zero

$$\left.\begin{array}{l}
u_R^4\langle n,0\left|\ell_R\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^3v_R\langle n,0\left|\ell_R\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2v_R^2\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
v_R^4\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots
\end{array}\right] = 0$$

The following doesn't give a zero.

$$\left.\begin{array}{l}
u_R^3v_R\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots(n+1)(n+2)u_R^3v_R \\
u_R^3v_R\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots n(n+1)u_R^3v_R \\
u_Rv_R^3\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n(n+1)u_Rv_R^3 \\
u_R^3v_R\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots(n+1)^2u_R^3v_R \\
u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n^2u_Rv_R^3 \\
u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n(n-1)u_Rv_R^3
\end{array}\right]$$

(3.26)

For

$$aaa^+a = \left(u_R\ell_R + v_R\ell_R^+\right)\left(u_R\ell_R + v_R\ell_R^+\right)\left(u_R\ell_R^+ + v_R\ell_R\right)\left(u_R\ell_R + v_R\ell_R^+\right)$$

$$= \frac{\gamma}{4\alpha^2}\langle n,0\left|\begin{matrix} u_R^4\ell_R\ell_R\ell_R^+\ell_R + u_R^3v_R\ell_R\ell_R\ell_R^+\ell_R^+ + u_R^3v_R\ell_R\ell_R\ell_R\ell_R + u_R^2v_R^2\ell_R\ell_R\ell_R\ell_R^+ \\ + u_R^3v_R\ell_R\ell_R^+\ell_R^+\ell_R + u_R^2v_R^2\ell_R\ell_R^+\ell_R^+\ell_R^+ + u_R^2v_R^2\ell_R\ell_R^+\ell_R\ell_R + u_Rv_R^3\ell_R\ell_R^+\ell_R\ell_R^+ \\ + u_R^3v_R\ell_R^+\ell_R\ell_R^+\ell_R + u_R^2v_R^2\ell_R^+\ell_R\ell_R^+\ell_R^+ + u_R^2v_R^2\ell_R^+\ell_R\ell_R\ell_R + u_Rv_R^3\ell_R^+\ell_R\ell_R\ell_R^+ \\ + u_R^2v_R^2\ell_R^+\ell_R^+\ell_R^+\ell_R + u_Rv_R^3\ell_R^+\ell_R^+\ell_R^+\ell_R^+ + u_Rv_R^3\ell_R^+\ell_R^+\ell_R\ell_R + v_R^4\ell_R^+\ell_R^+\ell_R\ell_R^+ \end{matrix}\right| n,0\rangle$$

The following gives zero.

$$\left.\begin{matrix} u_R^4\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0\left|\ell_R\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0\left|\ell_R\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ v_R^4\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \end{matrix}\right] = 0$$

The following doesn't give a zero

$$\left.\begin{array}{l}
u_R^3 v_R \langle n,0 | \ell_R \ell_R \ell_R^+ \ell_R^+ | n,0\rangle \ldots\ldots\ldots\ldots (n+1)(n+2) u_R^3 v_R \\
u_R v_R^3 \langle n,0 | \ell_R \ell_R^+ \ell_R \ell_R^+ | n,0\rangle \ldots\ldots\ldots\ldots (n+1)^2 u_R v_R^3 \\
u_R^3 v_R \langle n,0 | \ell_R^+ \ell_R \ell_R^+ \ell_R | n,0\rangle \ldots\ldots\ldots\ldots n^2 u_R^3 v_R \\
u_R v_R^3 \langle n,0 | \ell_R^+ \ell_R \ell_R \ell_R^+ | n,0\rangle \ldots\ldots\ldots\ldots n(n+1) u_R v_R^3 \\
u_R^2 v_R^2 \langle n,0 | \ell_R^+ \ell_R \ell_R^+ \ell_R | n,0\rangle \ldots\ldots\ldots\ldots n^2 u_R^2 v_R^2 \\
u_R v_R^3 \langle n,0 | \ell_R^+ \ell_R^+ \ell_R \ell_R | n,0\rangle \ldots\ldots\ldots\ldots n(n-1) u_R v_R^3
\end{array}\right] \tag{3.27}$$

For

$$aaa^+a^+ = \left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R^+ + v_R \ell_R\right)\left(u_R \ell_R^+ + v_R \ell_R\right)$$

$$= \frac{\gamma}{4\alpha^2} \langle n,0 \left| \begin{array}{l}
u_R^4 \ell_R \ell_R \ell_R^+ \ell_R^+ + u_R^3 v_R \ell_R \ell_R \ell_R^+ \ell_R + u_R^3 v_R \ell_R \ell_R \ell_R \ell_R^+ + u_R^2 v_R^2 \ell_R \ell_R \ell_R \ell_R \\
+ u_R^3 v_R \ell_R \ell_R^+ \ell_R^+ \ell_R^+ + u_R^2 v_R^2 \ell_R \ell_R^+ \ell_R^+ \ell_R + u_R^2 v_R^2 \ell_R \ell_R^+ \ell_R \ell_R^+ + u_R v_R^3 \ell_R \ell_R^+ \ell_R \ell_R \\
+ u_R^3 v_R \ell_R^+ \ell_R \ell_R^+ \ell_R^+ + u_R^2 v_R^2 \ell_R^+ \ell_R \ell_R^+ \ell_R + u_R^2 v_R^2 \ell_R^+ \ell_R \ell_R \ell_R^+ + u_R v_R^3 \ell_R^+ \ell_R \ell_R \ell_R \\
+ u_R^2 v_R^2 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ + u_R v_R^3 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R + u_R v_R^3 \ell_R^+ \ell_R^+ \ell_R \ell_R^+ + v_R^4 \ell_R^+ \ell_R^+ \ell_R \ell_R
\end{array} \right| n,0 \rangle$$

The following gives a zero

$$\left.\begin{array}{l}
u_R^3 v_R \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R \ell_R \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R \ell_R \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R^+ \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots
\end{array}\right] = 0$$

The following doesn't give a zero.

$$\left.\begin{array}{l}
u_R^4 \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots (n+1)(n+2) u_R^4 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n+1) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots (n+1)^2 u_R^2 v_R^2 \\
v_R^4 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n-1) v_R^4 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n^2 u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n+1) u_R^2 v_R^2
\end{array}\right]$$

(3.28)

For

$$aa^+aa = \left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R^+ + v_R \ell_R\right)\left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R + v_R \ell_R^+\right)$$

$$=\frac{\gamma}{4\alpha^2}\langle n,0\left|\begin{matrix} u_R^4\ell_R\ell_R^+\ell_R\ell_R+u_R^3v_R\ell_R\ell_R^+\ell_R\ell_R^++u_R^3v_R\ell_R\ell_R^+\ell_R^+\ell_R+u_R^2v_R^2\ell_R\ell_R^+\ell_R^+\ell_R^+ \\ +u_R^3v_R\ell_R\ell_R\ell_R\ell_R+u_R^2v_R^2\ell_R\ell_R\ell_R\ell_R^++u_R^2v_R^2\ell_R\ell_R\ell_R^+\ell_R+u_Rv_R^3\ell_R\ell_R\ell_R^+\ell_R^+ \\ +u_R^3v_R\ell_R^+\ell_R^+\ell_R\ell_R+u_R^2v_R^2\ell_R^+\ell_R^+\ell_R\ell_R^++u_R^2v_R^2\ell_R^+\ell_R^+\ell_R^+\ell_R+u_Rv_R^3\ell_R^+\ell_R^+\ell_R^+\ell_R^+ \\ +u_R^2v_R^2\ell_R^+\ell_R\ell_R\ell_R+u_Rv_R^3\ell_R^+\ell_R\ell_R\ell_R^++u_Rv_R^3\ell_R^+\ell_R\ell_R^+\ell_R+v_R^4\ell_R^+\ell_R\ell_R^+\ell_R^+ \end{matrix}\right| n,0\rangle$$

The following gives a zero.

$$\left.\begin{matrix} v_R^4\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0\left|\ell_R\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ v_R^4\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \end{matrix}\right]=0$$

The following doesn't give a zero.

$$
\left.\begin{array}{l}
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots (n+1)^2 u_R^2 v_R^2 \\
u_R^3 v_R \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n+1) u_R^3 v_R \\
u_R^3 v_R \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots (n+1)(n+2) u_R^3 v_R \\
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n-1) u_R^3 v_R \\
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n+1) u_R^3 v_R \\
u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n^2 u_R v_R^3
\end{array}\right] \tag{3.29}
$$

For

$$
aa^+aa^+ = \left(u_R\ell_R + v_R\ell_R^+\right)\left(u_R\ell_R^+ + v_R\ell_R\right)\left(u_R\ell_R + v_R\ell_R^+\right)\left(u_R\ell_R^+ + v_R\ell_R\right)
$$

$$
= \frac{\gamma}{4\alpha^2} \langle n,0 \left| \begin{array}{l}
u_R^4 \ell_R \ell_R^+ \ell_R \ell_R^+ + u_R^3 v_R \ell_R \ell_R^+ \ell_R \ell_R + u_R^3 v_R \ell_R \ell_R^+ \ell_R^+ \ell_R^+ + u_R^2 v_R^2 \ell_R \ell_R^+ \ell_R^+ \ell_R \\
+ u_R^3 v_R \ell_R \ell_R \ell_R \ell_R^+ + u_R^2 v_R^2 \ell_R \ell_R \ell_R \ell_R + u_R^2 v_R^2 \ell_R \ell_R \ell_R^+ \ell_R^+ + u_R v_R^3 \ell_R \ell_R \ell_R^+ \ell_R \\
+ u_R^3 v_R \ell_R^+ \ell_R^+ \ell_R \ell_R^+ + u_R^2 v_R^2 \ell_R^+ \ell_R^+ \ell_R \ell_R + u_R^2 v_R^2 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ + u_R v_R^3 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R \\
+ u_R^2 v_R^2 \ell_R^+ \ell_R \ell_R \ell_R^+ + u_R v_R^3 \ell_R^+ \ell_R \ell_R \ell_R + u_R v_R^3 \ell_R^+ \ell_R \ell_R^+ \ell_R^+ + v_R^4 \ell_R^+ \ell_R \ell_R^+ \ell_R
\end{array} \right| n,0 \rangle
$$

The following gives a zero

$$\left.\begin{array}{l}
u_R^3 v_R \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R \ell_R \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots
\end{array}\right] = 0$$

The following does not give a zero

$$\left.\begin{array}{l}
u_R^4 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots (n+1)^2 u_R^4 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots n(n+1) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots (n+1)(n+2) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots n(n-1) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots n(n+1) u_R^2 v_R^2 \\
v_R^4 \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots n^2 v_R^4
\end{array}\right]$$

(3.30)

For

$$aa^+a^+a = \left(u_R\ell_R + v_R\ell_R^+\right)\left(u_R\ell_R^+ + v_R\ell_R\right)\left(u_R\ell_R^+ + v_R\ell_R\right)\left(u_R\ell_R + v_R\ell_R^+\right)$$

$$=\frac{\gamma}{4\alpha^2}\langle n,0\left|\begin{array}{l} u_R^4\ell_R\ell_R^+\ell_R^+\ell_R+u_R^3v_R\ell_R\ell_R^+\ell_R^+\ell_R^++u_R^3v_R\ell_R\ell_R^+\ell_R\ell_R+u_R^2v_R^2\ell_R\ell_R^+\ell_R\ell_R^+ \\ +u_R^3v_R\ell_R\ell_R\ell_R^+\ell_R+u_R^2v_R^2\ell_R\ell_R\ell_R^+\ell_R^++u_R^2v_R^2\ell_R\ell_R\ell_R\ell_R+u_Rv_R^3\ell_R\ell_R\ell_R\ell_R^+ \\ +u_R^3v_R\ell_R^+\ell_R^+\ell_R^+\ell_R+u_R^2v_R^2\ell_R^+\ell_R^+\ell_R\ell_R+u_R^2v_R^2\ell_R^+\ell_R^+\ell_R^+\ell_R^++u_Rv_R^3\ell_R^+\ell_R^+\ell_R\ell_R^+ \\ +u_R^2v_R^2\ell_R^+\ell_R\ell_R^+\ell_R+u_Rv_R^3\ell_R^+\ell_R\ell_R\ell_R+u_Rv_R^3\ell_R^+\ell_R\ell_R^+\ell_R^++v_R^4\ell_R^+\ell_R\ell_R\ell_R^+ \end{array}\right| n,0\rangle$$

The following gives zero

$$\left.\begin{array}{l} u_R^3v_R\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_Rv_R^3\langle n,0\left|\ell_R\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \end{array}\right] = 0$$

The following does not give a zero

$$\left.\begin{array}{l}
u_R^4 \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n+1) u_R^4 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots (n+1)^2 u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots (n+1)(n+2) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n-1) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n^2 u_R^2 v_R^2 \\
v_R^4 \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n+1) v_R^4
\end{array}\right] \tag{3.31}$$

For

$$aa^+a^+a^+ = \left(u_R\ell_R + v_R\ell_R^+\right)\left(u_R\ell_R^+ + v_R\ell_R\right)\left(u_R\ell_R^+ + v_R\ell_R\right)\left(u_R\ell_R^+ + v_R\ell_R\right)$$

$$= \frac{\gamma}{4\alpha^2} \langle n,0 \left| \begin{array}{l}
u_R^4 \ell_R \ell_R^+ \ell_R^+ \ell_R^+ + u_R^3 v_R \ell_R \ell_R^+ \ell_R^+ \ell_R + u_R^3 v_R \ell_R \ell_R^+ \ell_R \ell_R^+ + u_R^2 v_R^2 \ell_R \ell_R^+ \ell_R \ell_R \\
+u_R^3 v_R \ell_R \ell_R \ell_R^+ \ell_R^+ + u_R^2 v_R^2 \ell_R \ell_R \ell_R^+ \ell_R + u_R^2 v_R^2 \ell_R \ell_R \ell_R \ell_R^+ + u_R v_R^3 \ell_R \ell_R \ell_R \ell_R \\
+u_R^3 v_{RR}^{\ 2} \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ + u_R^2 v_R^2 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R + u_R^2 v \ell_R^+ \ell_R^+ \ell_R \ell_R^+ + u_R v_R^3 \ell_R^+ \ell_R^+ \ell_R \ell_R \\
+u_R^2 v_R^2 \ell_R^+ \ell_R \ell_R^+ \ell_R^+ + u_R v_R^3 \ell_R^+ \ell_R \ell_R \ell_R^+ + u_R v_R^3 \ell_R^+ \ell_R \ell_R^+ \ell_R + v_R^4 \ell_R^+ \ell_R \ell_R \ell_R
\end{array} \right| n,0 \rangle$$

The following gives zero

$$\left.\begin{array}{l}
u_R^4\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2v_R^2\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2v_R^2\langle n,0\left|\ell_R\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_Rv_R^3\langle n,0\left|\ell_R\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^3v_R\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
v_R^4\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots
\end{array}\right] = 0$$

The following does not give a zero.

$$\left.\begin{array}{l}
u_R^3v_R\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n(n+1)u_R^3v_R \\
u_R^3v_R\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots (n+1)^2u_R^3v_R \\
u_R^3v_R\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots (n+1)(n+2)u_R^3v_R \\
u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n(n-1)u_Rv_R^3 \\
u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n^2u_Rv_R^3 \\
u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots n(n+1)u_Rv_R^3
\end{array}\right] \tag{3.32}$$

For

$$a^+aaa=\left(u_R\ell_R^++v_R\ell_R\right)\left(u_R\ell_R+v_R\ell_R^+\right)\left(u_R\ell_R+v_R\ell_R^+\right)\left(u_R\ell_R+v_R\ell_R^+\right)$$

$$=\frac{\gamma}{4\alpha^2}\langle n,0\left|\begin{matrix}u_R^4\ell_R^+\ell_R\ell_R\ell_R+u_R^3v_R\ell_R^+\ell_R\ell_R\ell_R^++u_R^3v_R\ell_R^+\ell_R\ell_R^+\ell_R+u_R^2v_R^2\ell_R^+\ell_R\ell_R^+\ell_R^+\\+u_R^3v_R\ell_R^+\ell_R^+\ell_R\ell_R+u_R^2v_R^2\ell_R^+\ell_R^+\ell_R\ell_R^++u_R^2v_R^2\ell_R^+\ell_R^+\ell_R^+\ell_R+u_Rv_R^3\ell_R^+\ell_R^+\ell_R^+\ell_R^+\\+u_R^3v_R\ell_R\ell_R\ell_R\ell_R+u_R^2v_R^2\ell_R\ell_R\ell_R\ell_R^++u_R^2v_R^2\ell_R\ell_R\ell_R^+\ell_R+u_Rv_R^3\ell_R\ell_R\ell_R^+\ell_R^+\\+u_R^2v_R^2\ell_R\ell_R^+\ell_R\ell_R+u_Rv_R^3\ell_R\ell_R^+\ell_R\ell_R^++u_Rv_R^3\ell_R\ell_R^+\ell_R^+\ell_R+v_R^4\ell_R\ell_R^+\ell_R^+\ell_R^+\end{matrix}\right|n,0\rangle$$

The following gives a zero.

$$\left.\begin{matrix}u_R^4\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots\\u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots\\u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots\\u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots\\u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots\\u_R^3v_R\langle n,0\left|\ell_R\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots\\u_R^2v_R^2\langle n,0\left|\ell_R\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots\\u_R^2v_R^2\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots\\u_R^2v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots\\v_R^4\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots\end{matrix}\right]=0$$

The following doesn't give a zero.

$$\left.\begin{array}{l}
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots n\left(n+1\right) u_R^3 v_R \\
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots n^2 u_R^3 v_R \\
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots n\left(n-1\right) u_R^3 v_R \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \left(n+1\right)\left(n+2\right) u_R v_R^3 \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \left(n+1\right)^2 u_R v_R^3 \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots n\left(n+1\right) u_R v_R^3
\end{array}\right]$$

(3.33)

For

$$a^+ a a a^+ = \left(u_R \ell_R^+ + v_R \ell_R\right)\left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R^+ + v_R \ell_R\right)$$

$$= \frac{\gamma}{4\alpha^2} \langle n,0 \left| \begin{array}{l}
u_R^4 \ell_R^+ \ell_R \ell_R \ell_R^+ + u_R^3 v_R \ell_R^+ \ell_R \ell_R \ell_R + u_R^3 v_R \ell_R^+ \ell_R \ell_R^+ \ell_R^+ + u_R^2 v_R^2 \ell_R^+ \ell_R \ell_R^+ \ell_R \\
+ u_R^3 v_R \ell_R^+ \ell_R^+ \ell_R \ell_R^+ + u_R^2 v_R^2 \ell_R^+ \ell_R^+ \ell_R \ell_R + u_R^2 v_R^2 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ + u_R v_R^3 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R \\
+ u_R^3 v_R \ell_R \ell_R \ell_R \ell_R^+ + u_R^2 v_R^2 \ell_R \ell_R \ell_R \ell_R + u_R^2 v_R^2 \ell_R \ell_R \ell_R^+ \ell_R^+ + u_R v_R^3 \ell_R \ell_R \ell_R^+ \ell_R \\
+ u_R^2 v_R^2 \ell_R \ell_R^+ \ell_R \ell_R^+ + u_R v_R^3 \ell_R \ell_R^+ \ell_R \ell_R + u_R v_R^3 \ell_R \ell_R^+ \ell_R^+ \ell_R^+ + v_R^4 \ell_R \ell_R^+ \ell_R^+ \ell_R
\end{array} \right| n,0\rangle$$

The following gives zero

$$\left.\begin{array}{l}
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R \ell_R \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots
\end{array}\right] = 0$$

The following does not give a zero.

$$\left.\begin{array}{l}
u_R^4 \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots n(n+1)u_R^4 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots n^2 u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots n(n-1)u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots (n+1)(n+2)u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots (n+1)^2 u_R^2 v_R^2 \\
v_R^4 \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots n(n+1)v_R^4
\end{array}\right]$$

(3.34)

For

$$a^+ a a^+ a = \left(u_R \ell_R^+ + v_R \ell_R\right)\left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R^+ + v_R \ell_R\right)\left(u_R \ell_R + v_R \ell_R^+\right)$$

$$=\frac{\gamma}{4\alpha^2}\langle n,0\left|\begin{matrix}u_R^4\ell_R^+\ell_R\ell_R^+\ell_R+u_R^3v_R\ell_R^+\ell_R\ell_R^+\ell_R^++u_R^3v_R\ell_R^+\ell_R\ell_R\ell_R+u_R^2v_R^2\ell_R^+\ell_R\ell_R\ell_R^+\\+u_R^3v_R\ell_R^+\ell_R^+\ell_R^+\ell_R+u_R^2v_R^2\ell_R^+\ell_R^+\ell_R^+\ell_R^++u_R^2v_R^2\ell_R^+\ell_R^+\ell_R\ell_R+u_Rv_R^3\ell_R^+\ell_R^+\ell_R\ell_R^+\\+u_R^3v_R\ell_R\ell_R\ell_R^+\ell_R+u_R^2v_R^2\ell_R\ell_R\ell_R^+\ell_R^++u_R^2v_R^2\ell_R\ell_R\ell_R\ell_R+u_Rv_R^3\ell_R\ell_R\ell_R\ell_R^+\\+u_R^2v_R^2\ell_R\ell_R^+\ell_R^+\ell_R+u_Rv_R^3\ell_R\ell_R^+\ell_R^+\ell_R^++u_Rv_R^3\ell_R\ell_R^+\ell_R\ell_R+v_R^4\ell_R\ell_R^+\ell_R\ell_R^+\end{matrix}\right|n,0\rangle$$

The following gives zero

$$\left.\begin{matrix}u_R^3v_R\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\\u_R^3v_R\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\\u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\\u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\\u_Rv_R^3\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\\u_R^3v_R\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\\u_R^2v_R^2\langle n,0\left|\ell_R\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\\u_Rv_R^3\langle n,0\left|\ell_R\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\\u_Rv_R^3\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\\u_Rv_R^3\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\end{matrix}\right]=0$$

The following does not give a zero.

$$\left.\begin{array}{l}
u_R^4\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n^2u_R^4\\
u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots n\left(n+1\right)u_R^2v_R^2\\
u_R^2v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n\left(n-1\right)u_R^2v_R^2\\
u_R^2v_R^2\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots\left(n+1\right)\left(n+2\right)u_R^2v_R^2\\
u_R^2v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n\left(n+1\right)u_R^2v_R^2\\
v_R^4\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots\left(n+1\right)^2v_R^4
\end{array}\right]$$

(3.35)

For

$$a^+aa^+a^+=\left(u_R\ell_R^++v_R\ell_R\right)\left(u_R\ell_R+v_R\ell_R^+\right)\left(u_R\ell_R^++v_R\ell_R\right)\left(u_R\ell_R^++v_R\ell_R\right)$$

$$=\frac{\gamma}{4\alpha^2}\langle n,0\left|\begin{array}{l}
u_R^4\ell_R^+\ell_R\ell_R^+\ell_R+u_R^3v_R\ell_R^+\ell_R\ell_R^+\ell_R^++u_R^3v_R\ell_R^+\ell_R\ell_R\ell_R+u_R^2v_R^2\ell_R^+\ell_R\ell_R\ell_R^+\\
+u_R^3v_R\ell_R^+\ell_R^+\ell_R^+\ell_R+u_R^2v_R^2\ell_R^+\ell_R^+\ell_R^+\ell_R^++u_R^2v_R^2\ell_R^+\ell_R^+\ell_R\ell_R+u_Rv_R^3\ell_R^+\ell_R^+\ell_R\ell_R^+\\
+u_R^3v_R\ell_R\ell_R\ell_R^+\ell_R+u_R^2v_R^2\ell_R\ell_R\ell_R^+\ell_R^++u_R^2v_R^2\ell_R\ell_R\ell_R\ell_R+u_Rv_R^3\ell_R\ell_R\ell_R\ell_R^+\\
+u_R^2v_R^2\ell_R\ell_R^+\ell_R^+\ell_R+u_Rv_R^3\ell_R\ell_R^+\ell_R^+\ell_R^++u_Rv_R^3\ell_R\ell_R^+\ell_R\ell_R+v_R^4\ell_R\ell_R^+\ell_R\ell_R^+
\end{array}\right|n,0\rangle$$

The following gives zero

$$\left.\begin{array}{l}
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^3 v_R \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots
\end{array}\right] = 0$$

The following does not give a zero .

$$\left.\begin{array}{l}
u_R^4 \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots n^2 u_R^4 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots n(n+1) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots n(n-1) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots (n+1)(n+2) u_R^2 v_R^2 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R \right| n,0\rangle \ldots\ldots\ldots\ldots n(n+1) u_R^2 v_R^2 \\
v_R^4 \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R^+ \right| n,0\rangle \ldots\ldots\ldots\ldots (n+1)^2 v_R^4
\end{array}\right] \tag{3.36}$$

For

$$a^+ a^+ a a = \left(u_R \ell_R^+ + v_R \ell_R\right)\left(u_R \ell_R^+ + v_R \ell_R\right)\left(u_R \ell_R + v_R \ell_R^+\right)\left(u_R \ell_R + v_R \ell_R^+\right)$$

$$=\frac{\gamma}{4\alpha^2}\langle n,0\left|\begin{matrix} u_R^4\ell_R^+\ell_R^+\ell_R\ell_R + u_R^3 v_R\ell_R^+\ell_R^+\ell_R\ell_R^+ + u_R^3 v_R\ell_R^+\ell_R^+\ell_R^+\ell_R + u_R^2 v_R^2\ell_R^+\ell_R^+\ell_R^+\ell_R^+ \\ + u_R^3 v_R\ell_R^+\ell_R\ell_R\ell_R + u_R^2 v_R^2\ell_R^+\ell_R\ell_R\ell_R^+ + u_R^2 v_R^2\ell_R^+\ell_R\ell_R^+\ell_R + u_R v_R^3\ell_R^+\ell_R\ell_R^+\ell_R^+ \\ + u_R^3 v_R\ell_R\ell_R^+\ell_R\ell_R + u_R^2 v_R^2\ell_R\ell_R^+\ell_R\ell_R^+ + u_R^2 v_R^2\ell_R\ell_R^+\ell_R^+\ell_R + u_R v_R^3\ell_R\ell_R^+\ell_R^+\ell_R^+ \\ + u_R^2 v_R^2\ell_R\ell_R\ell_R\ell_R + u_R v_R^3\ell_R\ell_R\ell_R\ell_R^+ + u_R v_R^3\ell_R\ell_R\ell_R^+\ell_R + v_R^4\ell_R\ell_R\ell_R^+\ell_R^+ \end{matrix}\right| n,0\rangle$$

The following gives zero

$$\left.\begin{matrix} u_R^3 v_R\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3 v_R\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2 v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3 v_R\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R v_R^3\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3 v_R\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R v_R^3\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2 v_R^2\langle n,0\left|\ell_R\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R v_R^3\langle n,0\left|\ell_R\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R v_R^3\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \end{matrix}\right] = 0$$

The following does not give a zero

$$\left.\begin{matrix} u_R^4\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n(n-1)u_R^4 \\ u_R^2 v_R^2\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots n(n+1)u_R^2 v_R^2 \\ u_R^2 v_R^2\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n^2 u_R^2 v_R^2 \\ u_R^2 v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots (n+1)^2 u_R^2 v_R^2 \\ u_R^2 v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n(n+1)u_R^2 v_R^2 \\ v_R^4\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots (n+1)(n+2)v_R^4 \end{matrix}\right] \quad (3.37)$$

For

$$a^{+}a^{+}aa^{+}=\left(u_R\ell_R^{+}+v_R\ell_R\right)\left(u_R\ell_R^{+}+v_R\ell_R\right)\left(u_R\ell_R+v_R\ell_R^{+}\right)\left(u_R\ell_R^{+}+v_R\ell_R\right)$$

$$=\frac{\gamma}{4\alpha^2}\langle n,0\left|\begin{matrix} u_R^4\ell_R^{+}\ell_R^{+}\ell_R\ell_R^{+}+u_R^3v_R\ell_R^{+}\ell_R^{+}\ell_R\ell_R+u_R^3v_R\ell_R^{+}\ell_R^{+}\ell_R^{+}\ell_R^{+}+u_R^2v_R^2\ell_R^{+}\ell_R^{+}\ell_R^{+}\ell_R \\ +u_R^3v_R\ell_R^{+}\ell_R\ell_R\ell_R^{+}+u_R^2v_R^2\ell_R^{+}\ell_R\ell_R\ell_R+u_R^2v_R^2\ell_R^{+}\ell_R\ell_R^{+}\ell_R^{+}+u_Rv_R^3\ell_R^{+}\ell_R\ell_R^{+}\ell_R \\ +u_R^3v_R\ell_R\ell_R^{+}\ell_R\ell_R^{+}+u_R^2v_R^2\ell_R\ell_R^{+}\ell_R\ell_R+u_R^2v_R^2\ell_R\ell_R^{+}\ell_R^{+}\ell_R^{+}+u_Rv_R^3\ell_R\ell_R^{+}\ell_R^{+}\ell_R \\ +u_R^2v_R^2\ell_R\ell_R\ell_R\ell_R^{+}+u_Rv_R^3\ell_R\ell_R\ell_R\ell_R+u_Rv_R^3\ell_R\ell_R\ell_R^{+}\ell_R^{+}+v_R^4\ell_R\ell_R\ell_R^{+}\ell_R \end{matrix}\right|n,0\rangle$$

The following gives zero

$$\left.\begin{matrix} u_R^4\langle n,0\left|\ell_R^{+}\ell_R^{+}\ell_R\ell_R^{+}\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0\left|\ell_R^{+}\ell_R^{+}\ell_R^{+}\ell_R^{+}\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R^{+}\ell_R^{+}\ell_R^{+}\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R^{+}\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R^{+}\ell_R\ell_R^{+}\ell_R^{+}\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R\ell_R^{+}\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R\ell_R^{+}\ell_R^{+}\ell_R^{+}\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^2v_R^2\langle n,0\left|\ell_R\ell_R\ell_R\ell_R^{+}\right|n,0\rangle\ldots\ldots\ldots\ldots \\ u_Rv_R^3\langle n,0\left|\ell_R\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\ v_R^4\langle n,0\left|\ell_R\ell_R\ell_R^{+}\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \end{matrix}\right]=0$$

The following does not give a zero

$$\left.\begin{array}{l}
u_R^3 v_R \langle n,0 \left| \ell_R^+ \ell_R^+ \ell_R \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n-1) u_R^4 \\
u_R^2 v_R^2 \langle n,0 \left| \ell_R^+ \ell_R \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n+1) u_R^2 v_R^2 \\
u_R v_R^3 \langle n,0 \left| \ell_R^+ \ell_R \ell_R^+ \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n^2 u_R^2 v_R^2 \\
u_R^3 v_R \langle n,0 \left| \ell_R \ell_R^+ \ell_R \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots (n+1)^2\, u_R^2 v_R^2 \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R^+ \ell_R^+ \ell_R \right| n,0 \rangle \ldots\ldots\ldots\ldots n(n+1) u_R^2 v_R^2 \\
u_R v_R^3 \langle n,0 \left| \ell_R \ell_R \ell_R^+ \ell_R^+ \right| n,0 \rangle \ldots\ldots\ldots\ldots (n+1)(n+2) v_R^4
\end{array}\right] \tag{3.38}$$

For $a^+ a^+ a^+ a = \left(u_R \ell_R^+ + v_R \ell_R\right)\left(u_R \ell_R^+ + v_R \ell_R\right)\left(u_R \ell_R^+ + v_R \ell_R\right)\left(u_R \ell_R + v_R \ell_R^+\right)$

$$= \frac{\gamma}{4\alpha^2} \langle n,0 \left| \begin{array}{l}
u_R^4 \ell_R^+ \ell_R^+ \ell_R^+ \ell_R + u_R^3 v_R \ell_R^+ \ell_R^+ \ell_R^+ \ell_R^+ + u_R^3 v_R \ell_R^+ \ell_R^+ \ell_R \ell_R + u_R^2 v_R^2 \ell_R^+ \ell_R^+ \ell_R \ell_R^+ \\
+ u_R^3 v_R \ell_R^+ \ell_R \ell_R^+ \ell_R + u_R^2 v_R^2 \ell_R^+ \ell_R \ell_R^+ \ell_R^+ + u_R^2 v_R^2 \ell_R^+ \ell_R \ell_R \ell_R + u_R v_R^3 \ell_R^+ \ell_R \ell_R \ell_R^+ \\
+ u_R^3 v_R \ell_R \ell_R^+ \ell_R^+ \ell_R + u_R^2 v_R^2 \ell_R \ell_R^+ \ell_R^+ \ell_R^+ + u_R^2 v_R^2 \ell_R \ell_R^+ \ell_R \ell_R + u_R v_R^3 \ell_R \ell_R^+ \ell_R \ell_R^+ \\
+ u_R^2 v_R^2 \ell_R \ell_R \ell_R^+ \ell_R + u_R v_R^3 \ell_R \ell_R \ell_R^+ \ell_R^+ + u_R v_R^3 \ell_R \ell_R \ell_R \ell_R + v_R^4 \ell_R \ell_R \ell_R \ell_R^+
\end{array} \right| n,0 \rangle$$

The following gives zero

$$\left.\begin{array}{l}
u_R^4\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^3 v_R\langle n,0\left|\ell_R^+\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2 v_R^2\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2 v_R^2\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2 v_R^2\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2 v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2 v_R^2\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R^2 v_R^2\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
u_R v_R^3\langle n,0\left|\ell_R\ell_R\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots \\
v_R^4\langle n,0\left|\ell_R\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots
\end{array}\right] = 0$$

The following does not give a zero

$$\left.\begin{array}{l}
u_R^3 v_R\langle n,0\left|\ell_R^+\ell_R^+\ell_R\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n(n-1)u_R^3 v_R \\
u_R^2 v_R^2\langle n,0\left|\ell_R^+\ell_R\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n^2 u_R^2 v_R^2 \\
u_R v_R^3\langle n,0\left|\ell_R^+\ell_R\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots n(n+1)u_R v_R^3 \\
u_R^3 v_R\langle n,0\left|\ell_R\ell_R^+\ell_R^+\ell_R\right|n,0\rangle\ldots\ldots\ldots\ldots n(n+1)u_R^3 v_R \\
u_R v_R^3\langle n,0\left|\ell_R\ell_R^+\ell_R\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots (n+1)^2 u_R v_R^3 \\
u_R v_R^3\langle n,0\left|\ell_R\ell_R\ell_R^+\ell_R^+\right|n,0\rangle\ldots\ldots\ldots\ldots (n+1)(n+2)u_R v_R^3
\end{array}\right] \tag{3.39}$$

For

$$a^+a^+a^+a^+ = \left(u_R\ell_R^+ + v_R\ell_R\right)\left(u_R\ell_R^+ + v_R\ell_R\right)\left(u_R\ell_R^+ + v_R\ell_R\right)\left(u_R\ell_R^+ + v_R\ell_R\right)$$

$$=\frac{\gamma}{4\alpha^2}\langle n,0\left|\begin{matrix}u_R^4\ell_R^+\ell_R^+\ell_R^+\ell_R^+ + u_R^3v_R\ell_R^+\ell_R^+\ell_R^+\ell_R + u_R^3v_R\ell_R^+\ell_R^+\ell_R\ell_R^+ + u_R^2v_R^2\ell_R^+\ell_R^+\ell_R\ell_R \\ +u_R^3v_R\ell_R^+\ell_R\ell_R^+\ell_R^+ + u_R^2v_R^2\ell_R^+\ell_R\ell_R^+\ell_R + u_R^2v_R^2\ell_R^+\ell_R\ell_R\ell_R^+ + u_Rv_R^3\ell_R^+\ell_R\ell_R\ell_R \\ +u_R^3v_R\ell_R\ell_R^+\ell_R^+\ell_R^+ + u_R^2v_R^2\ell_R\ell_R^+\ell_R^+\ell_R + u_R^2v_R^2\ell_R\ell_R^+\ell_R\ell_R^+ + u_Rv_R^3\ell_R\ell_R^+\ell_R\ell_R \\ +u_R^2v_R^2\ell_R\ell_R\ell_R^+\ell_R^+ + u_Rv_R^3\ell_R\ell_R\ell_R^+\ell_R + u_Rv_R^3\ell_R\ell_R\ell_R\ell_R^+ + v_R^4\ell_R\ell_R\ell_R\ell_R\end{matrix}\right|n,0\rangle$$

The following gives zero

$$\left.\begin{matrix}u_R^4\langle n,0|\ell_R^+\ell_R^+\ell_R^+\ell_R^+|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0|\ell_R^+\ell_R^+\ell_R^+\ell_R|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0|\ell_R^+\ell_R^+\ell_R\ell_R^+|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0|\ell_R^+\ell_R\ell_R^+\ell_R^+|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0|\ell_R^+\ell_R\ell_R\ell_R|n,0\rangle\ldots\ldots\ldots\ldots \\ u_R^3v_R\langle n,0|\ell_R\ell_R^+\ell_R^+\ell_R^+|n,0\rangle\ldots\ldots\ldots\ldots \\ u_Rv_R^3\langle n,0|\ell_R\ell_R^+\ell_R\ell_R|n,0\rangle\ldots\ldots\ldots\ldots \\ u_Rv_R^3\langle n,0|\ell_R\ell_R\ell_R^+\ell_R|n,0\rangle\ldots\ldots\ldots\ldots \\ u_Rv_R^3\langle n,0|\ell_R\ell_R\ell_R\ell_R^+|n,0\rangle\ldots\ldots\ldots\ldots \\ v_R^4\langle n,0|\ell_R\ell_R\ell_R\ell_R|n,0\rangle\ldots\ldots\ldots\ldots\end{matrix}\right]=0$$

The following does not give a zero

$$\left.\begin{matrix}u_R^2v_R^2\langle n,0|\ell_R^+\ell_R^+\ell_R\ell_R|n,0\rangle\ldots\ldots\ldots\ldots n(n-1)u_R^2v_R^2 \\ u_R^2v_R^2\langle n,0|\ell_R^+\ell_R\ell_R^+\ell_R|n,0\rangle\ldots\ldots\ldots\ldots n^2u_R^2v_R^2 \\ u_R^2v_R^2\langle n,0|\ell_R^+\ell_R\ell_R\ell_R^+|n,0\rangle\ldots\ldots\ldots\ldots n(n+1)u_R^2v_R^2 \\ u_R^2v_R^2\langle n,0|\ell_R\ell_R^+\ell_R^+\ell_R|n,0\rangle\ldots\ldots\ldots\ldots n(n+1)u_R^2v_R^2 \\ u_R^2v_R^2\langle n,0|\ell_R\ell_R^+\ell_R\ell_R^+|n,0\rangle\ldots\ldots\ldots\ldots (n+1)^2u_R^2v_R^2 \\ u_R^2v_R^2\langle n,0|\ell_R\ell_R\ell_R^+\ell_R^+|n,0\rangle\ldots\ldots\ldots\ldots (n+1)(n+2)u_R^2v_R^2\end{matrix}\right] \tag{3.40}$$

Summing up equations 3.25, 3.26, 3.27, 3.28, 3.29, 3.30, 3.31, 3.32, 3.33, 3.34, 3.35, 3.36, 3.37, 3.38, 3.39 and equation 3.40.

$$= \frac{\gamma}{4\alpha^2} \left| \begin{array}{l} u_R^4 \left[7n^2 + 6n + 3 \right] \\ + u_R^2 v_R^2 \left[44n^2 + 44n + 22 \right] \\ + u_R^3 v_R \left[19n^2 + 20n + 11 \right] \\ + u_R v_R^3 \left[19n^2 + 18n + 9 \right] \\ + v_R^4 \left[7n^2 + 8n + 4 \right] \end{array} \right| \langle n,0 | n,0 \rangle \tag{3.41}$$

$$v_R^4 = u_R^4 = v_R^2 u_R^2 = u_R^3 v_R = u_R v_R^3 = \frac{1}{4} \tag{3.42}$$

Inserting Eqn. 3.42into Eqn. 3.41 and summing, it gives

$$= \frac{\gamma}{16\alpha^2} \left(96n^2 + 96n + 49 \right) \tag{3.43}$$

$$\langle n,0 | \mathrm{H}' | n,0 \rangle = \frac{\beta}{\alpha^3 \sqrt{8}} (0) + \frac{\gamma}{16\alpha^2} \left(96n^2 + 96n + 49 \right) \tag{3.44}$$

$$E_n' = \frac{\gamma}{16\alpha^2} \left(96n^2 + 96n + 49 \right) \tag{3.45}$$

The energy of the system is calculated as

$$E_n = E_n^o + E_n' = \left(n + \frac{1}{2} \right) \hbar\omega + \frac{\gamma}{16\alpha^2} \left(96n^2 + 96n + 49 \right) \tag{3.46}$$

3.3 Heat Capacity and Transition Temperature

Heat capacity is expressed as,

$$C = \left(\frac{dE_n}{dT}\right) \tag{3.47}$$

E_n is multiplied by thermal activation factor $e^{-\frac{\hbar\omega}{\kappa T}}$ to get $E = E_n e^{-\frac{\hbar\omega}{\kappa T}}$. This value of the energy will be used to get C, T_c etc.

$$C = \left(\frac{d}{dT}\left[Ee^{-\frac{\hbar\omega}{\kappa T}}\right]\right) \tag{3.48}$$

$$C = E\frac{\hbar\omega}{\kappa T^2}e^{-\frac{\hbar\omega}{\kappa T}} \tag{3.49}$$

3.4 Entropy

$$ds = \frac{dQ}{dT};$$

$$\int ds = \int\frac{dQ}{dT} = \int\frac{MCdT}{T} \tag{3.50}$$

$$s = \frac{M\hbar\omega E}{K}\int\frac{1}{T^3}e^{-\frac{\hbar\omega}{\kappa T}}dT \tag{3.51}$$

Integrating Eqn. 3.51 by parts we get

$$S = \frac{M\hbar\omega E}{\kappa}\left[\frac{\kappa}{T\hbar\omega} + \frac{3\kappa^2}{\hbar^2\omega^2} + \frac{6T\kappa^3}{\hbar^3\omega^3} + \frac{6T^2\kappa^4}{\hbar^4\omega^4}\right]e^{-\frac{\hbar\omega}{\kappa T}} \tag{3.52}$$

3.5 Binding Energy per Nucleon

The binding energy per nucleon or the binding fraction *f is* derived from Eq. (3.46) by dividing the total energy of the system by the number of nucleons in the system. This yield,

$$f = \frac{1}{A}\left(\left(n+\frac{1}{2}\right)\hbar\omega+\frac{\gamma}{16\alpha^2}\left(96n^2+96n+49\right)\right) \qquad (3.53)$$

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.1 Introduction

From Eq (3.46) the energy of the system can be calculated by the corresponding parameters $\alpha^2 = \frac{\mu\omega}{\hbar}$ and $\gamma = \frac{\hbar\omega}{a_o}$, For ground state energy the value of n=0 such that the Eq (3.46) reduces to

$$E_{(n=o)} = \frac{1}{2}\hbar\omega + \frac{49\gamma}{16\alpha^2} \tag{5.1}$$

The ground energy $E_{(n=0)}$ is used to calculate heat capacity C and entropy S.

The following values for different physical quantities have been used

$h = 6.626\text{x}10^{-34}\text{JS}$

$\omega = 6\text{x}10^{9}\text{s}^{-1}$

$\kappa = 1.3807\text{x}10^{-23}\text{J/K}$

$\mu = 8.369\text{x}10^{-28}\text{Kg}$

$a_o = 1.3\text{x}10^{-15}\text{A}^{1/3}\text{M}$

4.2 Variation of Heat Capacity with Temperature

The equation

$$C = \frac{\hbar\omega}{\kappa T^2}\left(\left(\left(n+\frac{1}{2}\right)\hbar\omega + \frac{\gamma}{16\alpha^2}\left(96n^2+96n+49\right)\right)\right)e^{-\frac{\hbar\omega}{\kappa T}}$$

has been used to calculate the values of heat capacity by varying the temperature from 0.0200K to 0.400K. The data is recorded on table 4.1

Table 4.1: Variation of Heat Capacity with Temperature.

Temperature *T* (K)	**Heat Capacity *C* (J/K)**
0.0200	8.69808E-28
0.0400	2.91262E-25
0.0600	1.42695E-24
0.0800	2.66492E-24
0.1000	3.50393E-24
0.1200	3.93237E-24
0.1400	4.07063E-24
0.1600	4.03045E-24
0.1800	3.88962E-24
0.2000	3.69726E-24
0.2200	3.48295E-24
0.2400	3.26398E-24
0.2600	3.05009E-24
0.2800	2.84646E-24
0.3000	2.65557E-24
0.3200	2.47833E-24
0.3400	2.31469E-24
0.3600	2.16413E-24
0.3800	2.02585E-24
0.4000	1.89894E-24

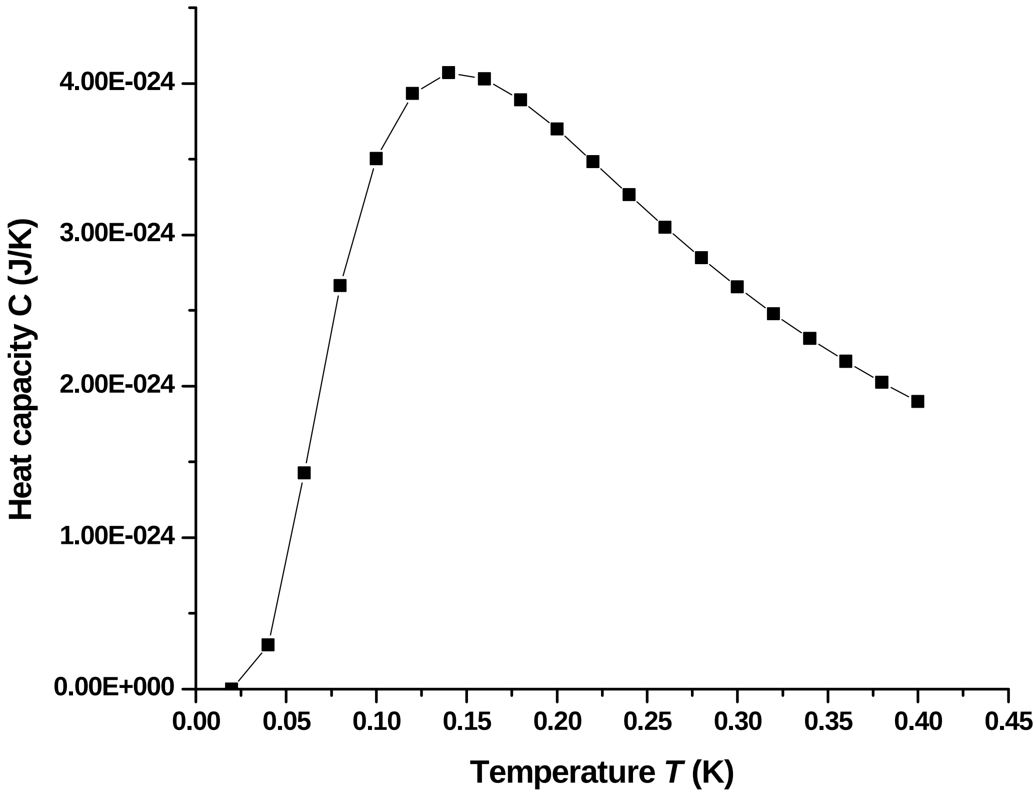

Figure 4.1: Variation of Heat Capacity with Temperature.

The relationship between heat capacity and temperature increases non-linearly to a maximum value corresponding to T=0.144K and decreases. This peak value at 0.144K, appears to be the transition temperature for ^{186}Re from normal to a superfluid state.

As T→ 0, the heat capacity converges to zero. This transition is of second order in which it occurs with no latent heat. The transition temperature is 0.144K this is in the region of other experimental values of 0.95K (Frieberthauser *et al.*, 1969). Previous experimental working with impure samples of Rhenium have reported superconducting transition at 2.2K and 2.4 K (Hulm *et al.*, 1957).

Table 4.2: Comparative values for transition Temperature.

	Theoretical values	Experimental values	Others
Transition Temperature T_c	0.144K	0.95K, 2.2K, 2.4K Frieberthauser, Hulm and Goodman	2.5-2.8K

There is a deviation in the theoretical transition temperature value from the experimental value, this is due to the strength of perturbation used in this study.

4.3 Variation of Excitation Energy with Heat Capacity at constant Temperature.

The equation

$$C = \frac{\hbar\omega}{\kappa T^2}\left(\left(\left(n+\frac{1}{2}\right)\hbar\omega + \frac{\gamma}{16\alpha^2}\left(96n^2+96n+49\right)\right)\right)e^{-\frac{\hbar\omega}{\kappa T}}$$

has been used to calculate the values of heat capacity by varying the excitation energy at different constant temperatures of 0.100K, 0.144K and 0.200K. The data is recorded on table 4.3

Table 4.3: Variation of Heat Capacity with Excitation Energy at constant Temperatures.

n	Excitation Energy (J)	Heat Capacity C (J/K)		
		T=0.1 K	T=0.144K	T=0.2K
0	2.17E-24	3.5E-24	4.07E-24	3.6972E-24
1	6.81E-24	1.1E-23	1.28E-23	1.1617E-23
2	1.22E-23	1.97E-23	2.28E-23	2.0737E-23
3	1.82E-23	2.94E-23	3.42E-23	3.1057E-23
4	2.5E-23	4.04E-23	4.69E-23	4.2577E-23
5	3.24E-23	5.24E-23	6.09E-23	5.5298E-23
6	4.06E-23	6.56E-23	7.63E-23	6.922E-23
7	4.94E-23	7.99E-23	9.29E-23	8.4341E-23
8	5.9E-23	9.54E-23	1.11E-22	1.0066E-22
9	6.93E-23	1.12E-22	1.3E-22	1.1819E-22
10	8.03E-23	1.3E-22	1.51E-22	1.3691E-22
11	9.19E-23	1.49E-22	1.73E-22	1.5683E-22
12	1.04E-22	1.69E-22	1.96E-22	1.7795E-22
13	1.17E-22	1.9E-22	2.21E-22	2.0028E-22
14	1.31E-22	2.12E-22	2.47E-22	2.238E-22
15	1.46E-22	2.36E-22	2.74E-22	2.4853E-22
16	1.61E-22	2.6E-22	3.02E-22	2.7445E-22
17	1.77E-22	2.86E-22	3.32E-22	3.0158E-22
18	1.93E-22	3.13E-22	3.64E-22	3.299E-22
19	2.11E-22	3.41E-22	3.96E-22	3.5943E-22
20	2.29E-22	3.7E-22	4.3E-22	3.9015E-22
21	2.47E-22	4E-22	4.65E-22	4.2208E-22
22	2.67E-22	4.31E-22	5.02E-22	4.5521E-22
23	2.87E-22	4.64E-22	5.39E-22	4.8954E-22
24	3.08E-22	4.98E-22	5.79E-22	5.2506E-22
25	3.29E-22	5.32E-22	6.19E-22	5.6179E-22
26	3.52E-22	5.68E-22	6.61E-22	5.9972E-22
27	3.75E-22	6.05E-22	7.04E-22	6.3885E-22
28	3.98E-22	6.44E-22	7.48E-22	6.7918E-22
29	4.22E-22	6.83E-22	7.94E-22	7.2071E-22
30	4.48E-22	7.24E-22	8.41E-22	7.6344E-22

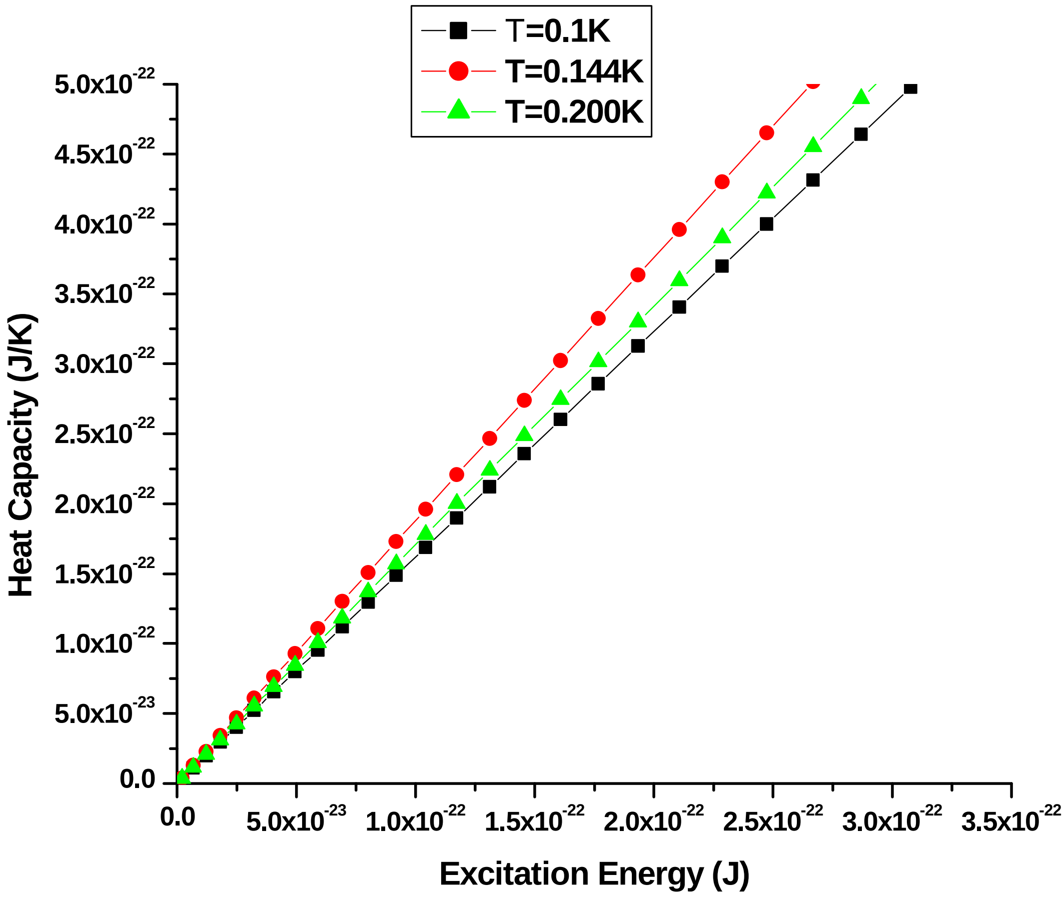

Figure 4.2: Variation of Heat Capacity with Excitation Energy at constant Temperatures.

The relationship between heat capacity and excitation energy in Fig. 4.2 has a linear relationship. The gradient increases with increases in temperature and attains a maximum gradient corresponding to the transition temperature (0.144K) and above this temperature the gradient lowers. This shows that heat capacity and excitation energy are maximum at transition temperature. The excitation energy approaches zero as the heat capacity goes to zero. These results suggests the strong dependence of the internal energy of the particles of the system on the heat capacity.

4.4: Variation of Entropy with Temperature

The equation

$$S = \frac{M\hbar\omega E}{\kappa}\left[\frac{\kappa}{T\hbar\omega} + \frac{3\kappa^2}{\hbar^2\omega^2} + \frac{6T\kappa^3}{\hbar^3\omega^3} + \frac{6T^2\kappa^4}{\hbar^4\omega^4}\right]e^{-\frac{\hbar\omega}{\kappa T}}$$

has been used to calculate the values of entropy by varying the temperature from 0.1K to 2.0 K. The data is recorded on table 4.4.

Table 4.4: Variation of Entropy with Temperature

Temperature T (K)	Entropy S (J/K)
0.1	5.18509E-22
0.2	1.04197E-21
0.3	1.57037E-21
0.4	2.10373E-21
0.5	2.64204E-21
0.6	3.18529E-21
0.7	3.73349E-21
0.8	4.28664E-21
0.9	4.84474E-21
1	5.40779E-21
1.1	5.97579E-21
1.2	6.54873E-21
1.3	7.12663E-21
1.4	7.70947E-21
1.5	8.29726E-21
1.6	8.89E-21
1.7	9.48769E-21
1.8	1.00903E-20
1.9	1.06979E-20
2	1.13104E-20

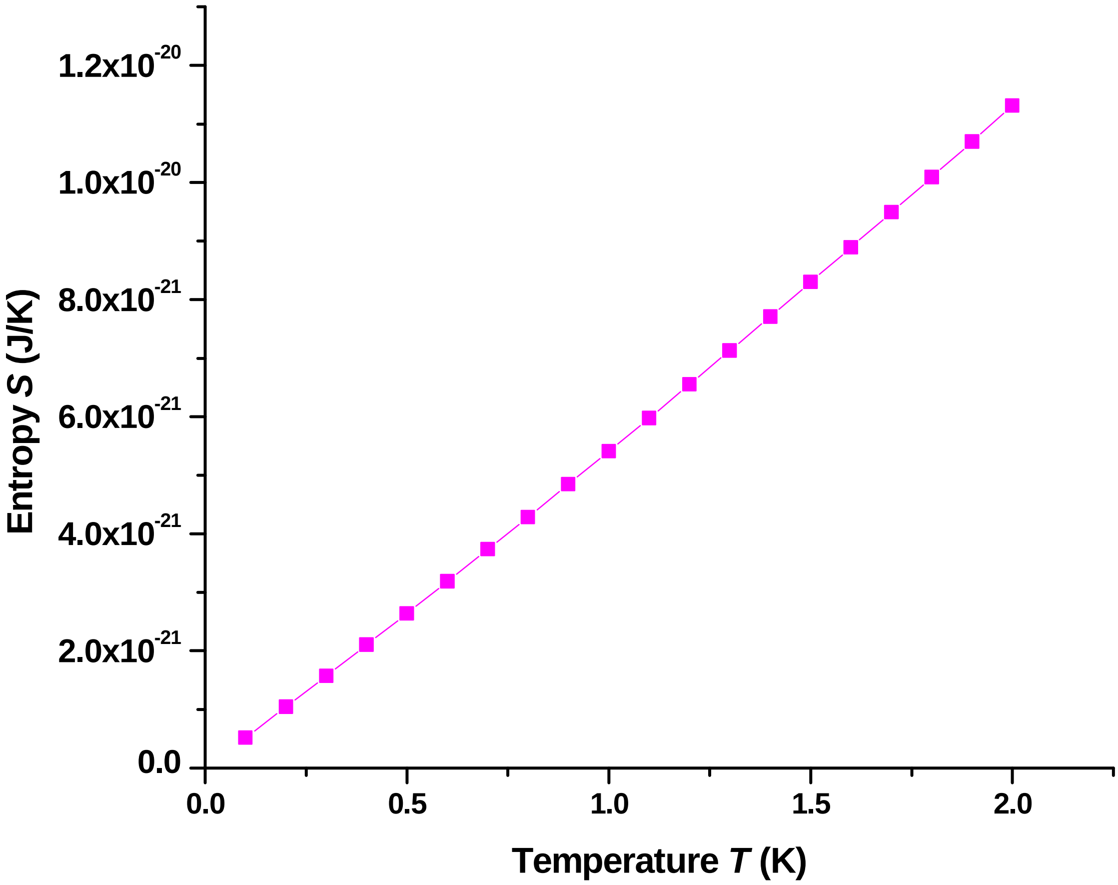

Figure 4.3: Variation of Entropy with Temperature.

The graph of entropy against temperature increases non-linearly with temperature. This is due to the fact that as the temperature increases, the nucleons are expected to be more disorderly this is in agreement with Dean *et al.*, 2003 results of entropy against temperature. For T $\rightarrow$ 0, the entropy converges to zero. This shows that Rhenium becomes orderly at very low temperatures.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

The second quantization approach in the study of the thermodynamic properties of $^{186}_{75}$Re has provided a basis to more precise values. It is now easy to compare some fundamental experimental facts of the properties of $^{186}_{75}$Re.

The energy of $^{186}_{75}$Re has been obtained successful, it has been used to calculate the heat capacity, Excitation energy and Entropy.

The heat capacity for $^{186}_{75}$Re exhibit *S*-shaped heat capacity curves as a function of temperature which is interpreted as a fingerprint of a phase transition (Hess *et al*., 2004) which is a second order phase transition (Belic *et al.,* 2004). We note that both theoretical and experimental results exhibit both *S*-shaped curves.

The values calculated for heat capacity are positive and this is mainly due to the Coulomb interaction. The relationship of heat capacity and temperature is non-linear, as temperature approaches zero heat capacity also converges to zero. These results obtained are similar to those reported by Dean *et al.,* 2003; Lui *et al*., 2001 and Khanna. *et al*., 2010 but within close temperature ranges. The transition temperature for $^{186}_{75}$Re of 0.144K with a peak value of 1.5J/K, suggests the possibility that the material attains a phase change to a superfluid state just below this transition temperature.

The transition temperature of 0.144K, from the theoretical calculations ascertains the results obtained from different experimental aspects. (Frieberthause and Notarys 1969). Some experimental working with impure samples of Rhenium have reported superfluid transition temperature at 0.95, 2.2 and 2.4 K (Hulm and Goodman 1957). And this could explain the deviations from the ideal theoretical results of 0.144K. The low T_c obtained could also be due to the strength of perturbation used for the calculations that may appear larger than the realistic perturbation such that to realize the superfluid state.

The calculated variation of heat capacity with excitation energy at constant temperature exhibits a linear relationship but with increasing gradients at higher values of constant temperatures to a maximum gradient corresponding to the critical temperature T_c. This could be strongly linked to thermal excitation of the nuclei and that the maximum heat capacity observed at different excited states corresponds to the critical transition temperature T_c.

From these results it can now be confirmed that it is possible to use the interaction potential and second quantization approach in the study of thermodynamics properties of the heavy nuclei and to predict accurately the transition temperature at which superfluidity appears in heavy nuclei.

5.2 Recommendations

1. Apart from second quantization approach other theoretical approaches may still be used to ascertain this result i.e statistical molecular dynamics.

2. There is need to consider an interaction potential that may account for Spin-dependent, density-dependent and velocity-dependent potential that may give more precise results.

3. Adding a perturbation to power six to improve on corrections.

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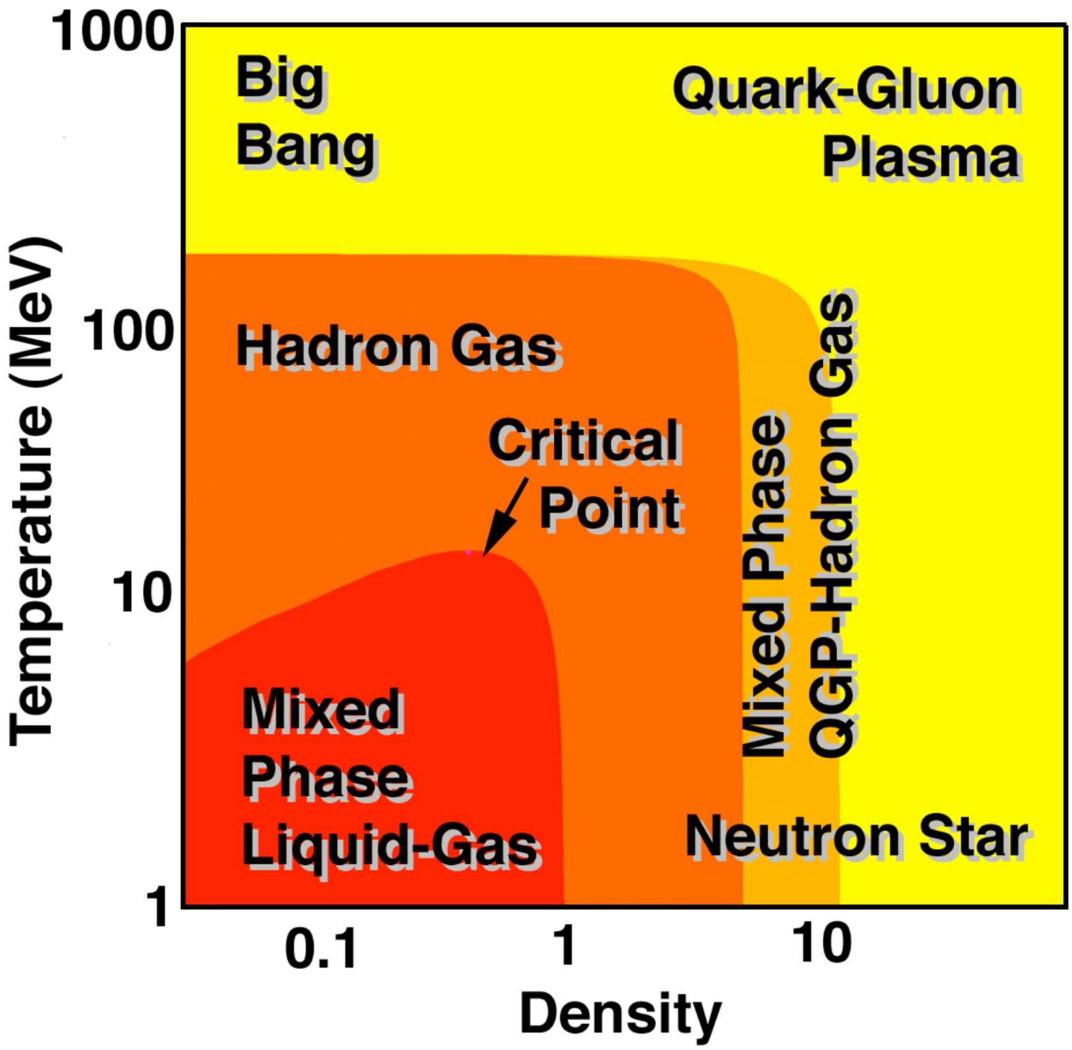
APPENDICES

Appendix I



The phase diagram for nuclear matter, as predicted theoretically. The horizontal axis shows the matter density, and the vertical axis shows the temperature. Both axes are shown in logarithmic scale, and the density is given in multiples of normal nuclear matter density. (Nuclear science-A guide to the nuclear science wall chart 2004 contemporary physics Education project [CPEP]