SOME NEW SECOND ORDER ROTATABLE DESIGNS BASED ON BALANCED INCOMPLETE BLOCK DESIGN

BY

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DECLARATION

Declaration by the student

I declare that this thesis is my original work and has never been done by any other person in any other institution

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SC/PGM/08/06

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Declaration by the supervisors

We confirm that the work reported in this thesis has been carried out by the candidate under our supervision and with our approval as the university supervisors.

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DEDICATION

To my children Dan, Joy and Abigael

ABSTRACT

The purpose of this study was to construct some new second order rotatable designs using balanced incomplete block designs. The study of rotatable designs is mainly concerned with the estimation of absolute response in designs and analysis of experiments. Second order rotatable designs in three, four, five, six and a generalization in k-factors (k>6) were constructed. The optimality criteria for these designs were also determined. To construct a second order rotatable design a BIBD (v, b, s, r, λ) was considered and a combination of points with unknown constants was taken and associated with 2^k factorial combinations of factors each at ± 1 levels to make level codes equidistant. This factorial combination form the set of points $s(a, a, ..., a)$ and $s(b, b, ..., o)$ .All such combinations form a design. Second order rotatable designs were obtained in three, four, five and six dimensions and finally a generalization (v=k) of the design in k-dimensions was obtained by derivation. An incidence matrix of BIBD was chosen suitably to satisfy the moment conditions of rotatability. In conclusion, some new second order rotatable designs in three, four, five and six factors and their generalization in k factors were obtained through balanced incomplete block designs and their optimalities determined.

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CHAPTER ONE

INTRODUCTION

1.1 Background to the study

Response surface methodology (RSM) is a collection of statistical and mathematical techniques useful for developing, improving and optimizing processes. The goal of most response surface research is to find a suitable approximating function for the purpose of predicting future response and to find levels of the input variables for which in some sense the response is optimized. The aim is to actually determine optimum operating conditions or to define a region in the space of the input variables where certain operating specifications are met.

A rotatable design is a series of response surface designs with the property that the variance of estimates of response at points equidistant from the centre of the design is constant. These designs ensure equal precision on the response estimates. The study of rotatable designs mainly emphasizes on the estimation of absolute response.

Rotatability of designs has been studied by a number of researchers, Box and Hunter (1957) introduced rotatable designs for the exploration of response surfaces. He emphasized the estimation of absolute response, Huda (1987) constructed third order rotatable design from design of lower dimension, Victorbabu (2006) constructed rotatable design using a pair of incomplete block design, Victorbabu and Vasundharadevi (2009) studied the efficiencies of second order response surface designs for the estimation of response and slopes using symmetrical unequal block arrangement two unequal block sizes. Some new third order rotatable design in five dimensions through balanced incomplete block (BIB) designs has also been suggested by Koske et al (2011). In this study some new second order rotatable designs are obtained through balanced incomplete block (BIB) designs. The method of construction will share some features proposed by Koske et al. (2011). Here we start by constructing second order rotatable designs with three factors then four, five and six factors and later give a generalization using k factors ($k>6$).

1.2 Balanced Incomplete Block (BIB) Designs

A balanced incomplete block (BIB) design denoted by (v, b, r, s, λ) is an arrangement of v-treatments in b-blocks each containing s (< v) treatments and satisfying the following conditions.

a) Every treatment occurs at most once in a block
b) Every treatment occurs in exactly r-blocks
c) Every pair of treatments occurs together in λ blocks

The quantities v, b, r, s and λ are called the parameters of a BIB design

The necessary conditions for the existence of a BIB design is ;-

(i) $rv = sb$

(ii) $(v-1) = r(s-1)$

(iii) $r > \lambda$

(iv) $b \geq v$

1.3 Second Order Rotatable Designs

Suppose we want to use a second response surface $D = x_{iu}$ to fit the surface

$$Y_u = b_o + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^{\ 2} + \sum_{i \prec j} \sum b_{ij} x_{iu} x_{ju} + \ell_u$$

Where x_{iu} denotes the level of the i^{th} factor ($i = 1,2,......v$ in the u^{th} run ($u = 1,2,......N$ of the experiments e_u 's are uncorrelated random errors with mean zeros and variance σ^2

b_0, b_i, b_{ii}, b_{ij} are the parameters of the model and y_u is the response observed at the u^{th} design point.

Then the surface is said to be a second order rotatable arrangement if it satisfies the following two moment conditions.

(i) $$\sum_{u=1}^{N} x_{iu}^{\ 2} = N\lambda_2 = A$$

(ii) $$\sum_{u=1}^{N} x_{iu}^{\ 4} = 3\sum_{u=1}^{N} x_{iu}^{\ 2} x_{ju}^{\ 2} = 3N\lambda_4 = 3B$$

Where λ_2 and λ_4 are constants,

And $A = \dfrac{N}{K}$, $B = \dfrac{N}{K(K+2)}$

A necessary condition for the existence of a non-singular second order design is

$$\frac{\lambda_4}{\lambda_2^{\;2}} > \frac{K}{K+2} \text{ i.e } \quad \frac{NB}{A^2} > \frac{K}{K+2}$$

1.4 Statement of Problem

The problem is to generalize the construction of second order rotatable design based on balanced incomplete block design (BIBD). A review of the necessary and sufficient conditions for a design to be second order rotatable is done and also the variances and the co-variances of the parameters estimates are reviewed.

1.5 Objectives

1.5.1 Main Objective

The main objective of the study is to construct some new second order rotatable designs using the balanced incomplete block design.

1.5.2 Specific Objectives

To meet the above main objective, we shall strive to attain the following specific objectives:

(i) To construct some new second order rotatable designs in three, four, five, six and k-dimensions.

(ii) To obtain optimality criteria for these designs.

1.6 Significance of the Study

The study would be desirable where the experimenter is interested in some $k-1$ subset of factors that would give rise to a BIB $(k,b,s,k-1,\lambda)$ and a replicate of the incidence matrix, where the second order rotatable design in k dimension will have $k-1$ dimension that involves the subset of factors desired by the experimenter.

This study would be used when carrying out experiments that require replications.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Response surface methodology (RSM) is a statistical technique very useful in design and analysis of experiments. It involves a dependent variable y_u such as yield and is called the response variable.

In general $y_u = f(x_{1u}, x_{2u}, ...x_{ku}) + e_u$ where $u = 1,2,\cdots\cdots.N$ represents the N-observations and x_{iu} is the level of the i^{th} factor in the u^{th} observation and y_u is the response, e_u is the random error with mean zero and variance σ^2. Response surface method is useful where several independent variables influence a dependent variable.

The concept of rotatability which is very important in response surface was proposed by Box and Hunter (1957). The study emphasizes on the estimation of absolute response. The k-dimensional point set forms a second order rotatable arrangement in k- factors if the following conditions hold

$$\sum_{u=1}^{N} x_{iu}^{2} = A \qquad i = (1,2\cdots\cdots k\ )$$

$$\sum_{u=1}^{N} x_{iu}^{4} = 3\sum_{u=1}^{N} x_{iu}^{2} x_{ju}^{2} = 3B \tag{1}$$

and all other sums of powers and products up to order four are zero i.e. $\sum_{u=1}^{N}\prod_{i=1}^{v} x_{iu}\alpha_u = 0$

if any α is odd for $\sum \alpha_i \leq 4$ and $A = N\lambda_2$, $B = N\lambda_4$

This arrangement of the points forms a non-singular second order rotatable designs if it satisfies the necessary condition of a second order rotatable non-singular design i.e. $\frac{NB}{A} > \frac{k}{k+2}$ which is the condition required for a second order arrangement of points to form a second order rotatable design ,where also

$$A = \frac{N}{K} \text{ and } B = \frac{N}{K(K+2)} \tag{2}$$

According to Das and Narasimham (1962) they constructed many third order rotatable designs by taking appropriate combinations of the symmetric point set or their suitably balanced subset obtained through balanced incomplete block designs and fractional replications. Park (1987) studied estimations of local slope (rate of change) of the response. He gave an example of rate of change in the yield of crop to various fertilizer doses. Huda (1987) constructed third order rotatable designs in k-dimensions from those in lower dimensions. Victorbabu and Narasimham (1991) constructed second order slope rotatable designs (SOSRD) using balanced incomplete block design. Das et.al. (1999) studied response surface design, symmetrical and asymmetrical rotatable an modified. Victorbabu (2005) studied modified slope rotatable central composite designs (SRCCD). Victorbabu and vasundharadevi (2005b) studied modified second order response surface designs using BIBD. Victorbabu (2006) studied second order rotatable designs using a pair of incomplete block designs. He considered a second order rotatable design by taking combinations with unknown constants and associated a 2^k factorial combinations or a suitable fraction of it with factors each at ± 1 levels to make the level codes equidistant to form a design. According to Victorbabu and Vasundharaderi (2009) a design for fitting a response surface consists of number of suitable combinations of levels of several input factors. They considered v factors and N combinations in the design each having a constant number of levels. A response surface design can be written as v-rows and N columns each. Each row being a combination of v- levels codes on from each of v- ordered factors. This combination of level codes he called a design point and the combination with 0- code for each factor is called a central point. They were considering the efficiencies of various second order response surface designs. Victorbabu (2009) also reviewed modified SOSRDs. He also presented different methods of construction of modified SOSRDs using central composite designs, Balanced incomplete block design, pairwise balanced designs symmetrical unequal block arrangement with two unequal block sizes. Victorbabu and Surekha (2011) constructed a SOSRD using balanced incomplete block design. The study of rotatable designs is mainly emphasized on the estimation of differences of yields and its precision. Estimation of differences in responses at two different points in the factor space is of great importance. Koske et al (2011) constructed a third order rotatable design five-dimensions from a third order rotatable design in lower dimensions through balanced incomplete block designs (BIBDs). VictorBabu (2011) explored a new method of construction of second order-slope-rotatable design using incomplete block designs with unequal block sizes. Victorbabu and Rajyalakshmi (2012) studied a new method of construction of robust second order rotatable designs using balanced incomplete block designs. Furthermore, Mutai, Koske and Mutiso (2012) constructed some new four dimensional third order rotatable design through balanced incomplete design.

The method used by Huda (1987) and Koske, et al (2011), (2012) can be used to obtain a generalized method of constructing designs. In this study some new second order rotatable designs using BIBD were constructed.

CHAPTER THREE

METHODOLOGY

3.1 Construction Methods

To construct a second order rotatable design a BIB$\left(v,b,s,r,\lambda\right)$ is considered and a combination of points with unknown constants is taken and associated with 2^k factorial combinations of factors each at ± 1 levels to make level codes equidistant. This factorial combination forms the set of points $s(a,a,...,a)$ and $s(b,b,...,o)$ .All such combinations form a design. Second order rotatable designs are obtained in three, four, five and six dimensions and finally a generalization (v=k) of the design in k-dimensions is obtained by derivation.

An incidence matrix of BIBD is chosen suitably to satisfy the moment conditions of rotatability, that is

(i) $\sum_{u=1}^{N} x_{iu}{}^{2} = N\lambda_2 = A$ and

(ii) $\sum_{u=1}^{N} x_{iu}{}^{4} = 3\sum_{u=1}^{N} x_{iu}{}^{2} x_{ju}{}^{2} = 3N\lambda_4 = 3B$

Where all other sums of odd powers and cross products up to order four are zeros, that is

$\sum_{u=1}^{N}\prod_{i=1}^{v} x_{iu}\alpha_i = 0$ if any α_i is odd for $\sum \alpha_i \leq 4$

These arrangements of points is said to form a rotatable design of second order only if it forms a non-singular second order design.

These points should give rise to a non-singular $X'X$ matrix, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_k \end{bmatrix}.$$

Let X' be the transpose of X and $N^{-1}(X'X)$ is the moment matrix M of the arrangement of N points in $k-$dimensional factor space.

The determinant of M is obtained and this gives the non-singularity conditions for a second order experimental design to be rotatable. Box and Hunter (1957) gives the necessary non-singularity condition as $\frac{\lambda_4}{\lambda_2^{\ 2}} > \frac{K}{K+2}$ and the arrangement forms a non-singular second order rotatable design. The strict inequality is achieved by addition of centre points.

3.2 Optimality Methods

The optimality criteria were also obtained that is the trace, eigen values and determinant.

The trace was obtained by adding all the elements in the principal diagonal of the matrix M. The Eigen values were given as $\left|\lambda I_{k+1} - M\right| = 0$ and the determinant of M is also obtained.

3.3 Review of Conditions of Rotatability

A review of the moment and non-singularity conditions is done where a given set of points is said to be second order rotatable arrangement if it satisfies the moment conditions $\sum_{u=1}^{N} x_{iu}^{\ 2} = N\lambda_2 = A$ and

$$\sum_{u=1}^{N} x_{iu}^{\ 4} = 3\sum_{u=1}^{N} x_{iu}^{\ 2} x_{ju}^{\ 2} = 3N\lambda_4 = 3B$$

The second order rotatble arrangement becomes a second order rotatable design where the arrangement of points also satisfy the non-singularity condition $\frac{\lambda_4}{\lambda_2^{\ 2}} \geq \frac{K}{K+2}$.

3.3.1 Non-singularity Conditions

These are the conditions that must be satisfied in order for a second order experimental design to be rotatable. From the moment matrix of second order its polynomial determinant is determined and this will give the non-singularity conditions. The response surface approximated is second order polynomial and is given by $y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=u}^{v} b_{ii} x_{iu}{}^2 + \sum_{i<j} \sum b_{ij} x_{iu} x_{ju}$ where $u = 1,2....,N$ (3)

Consider the matrix $\underline{X} = \begin{bmatrix} \underline{x_0} \\ \underline{x_1} \\ \underline{x_2} \\ . \\ . \\ x_k \end{bmatrix}$ then M is obtained as $\frac{1}{N} X'X$

$$
\left[\begin{array}{ccccccc|ccccccc|ccccccc}
x_0^2 & x_0^2 x_1^2 & x_0^2 x_2^2 & . & . & . & x_0^2 x_k^2 & & & & & & & & & & & & & \\
 & x_1^4 & x_1^2 x_2^2 & . & . & . & x_1^2 x_k^2 & & & & & & & & & & & & & \\
 & & x_2^4 & . & . & . & x_2^2 x_k^2 & & & & & & & & & & & & & \\
 & & & . & & & . & & & H & & & & & & I & & & & \\
 & & & & . & & . & & & & & & & & & & & & & \\
 & & & & & . & . & & & & & & & & & & & & & \\
 & & & & & & x_k^4 & & & & & & & & & & & & & \\
\hline
 & & & & & & & x_1^2 x_2^2 & x_1^2 x_2 x_3 & . & . & . & x_1^2 x_2 x_k & & & & & & & \\
 & & & & & & & & x_2^2 x_3^2 & . & . & . & x_2^2 x_3 x_k & & & & & & & \\
 & & & & & & & & & . & & & & & & J & & & & \\
 & & & & & & & & & & . & & & & & & & & & \\
 & & & & & & & & & & & . & & & & & & & & \\
 & & & & & & & & & & & & x_{k-1} x_k^2 & & & & & & & \\
\hline
 & & & & & & & & & & & & & x_1^2 & x_1 x_2 & . & . & . & x_1 x_k & \\
 & & & & & & & & & & & & & & x_2^2 & . & . & . & x_2 x_k & \\
 & & \mathit{Symmetric} & & & & & & & & & & & & & . & & & . & \\
 & & & & & & & & & & & & & & & & . & & . & \\
 & & & & & & & & & & & & & & & & & . & . & \\
 & & & & & & & & & & & & & & & & & & x_k^2 &
\end{array}\right] \qquad (4)
$$

which simplifies to

$$\left[\begin{array}{ccccccc:ccccc:ccccc}
1 & \lambda_2 & \lambda_2 & \cdot & \cdot & \cdot & \lambda_2 & & & & & & & & & & \\
 & 3\lambda_2 & \lambda_4 & \cdot & \cdot & \cdot & \lambda_4 & & & & & & & & & & \\
 & & 3\lambda_4 & \cdot & \cdot & \cdot & \lambda_4 & & & H & & & & & I & & \\
 & & & \cdot & & & \cdot & & & & & & & & & & \\
 & & & & \cdot & & \cdot & & & & & & & & & & \\
 & & & & & \cdot & \cdot & & & & & & & & & & \\
 & & & & & & 3\lambda_4 & & & & & & & & & & \\ \hdashline
 & & & & & & & \lambda_4 & 0 & \cdot\ \cdot\ \cdot & & 0 & & & & & \\
 & & & & & & & & \lambda_4 & \cdot\ \cdot\ \cdot & & 0 & & & & & \\
 & & & & & & & & & \cdot & & \cdot & & & J & & \\
 & & & & & & & & & & \cdot & \cdot & & & & & \\
 & & & & & & & & & & \cdot & \cdot & & & & & \\
 & & & & & & & & & & & \lambda_4 & & & & & \\ \hdashline
 & & & & & & & & & & & & \lambda_2 & 0 & \cdot\ \cdot\ \cdot & & 0 \\
 & & & & & & & & & & & & & \lambda_2 & \cdot\ \cdot\ \cdot & & 0 \\
 & & & Symmetric & & & & & & & & & & & \cdot & & \cdot \\
 & & & & & & & & & & & & & & & \cdot & \cdot \\
 & & & & & & & & & & & & & & & \cdot & \cdot \\
 & & & & & & & & & & & & & & & & \lambda_2
\end{array}\right]_{(k+1)\times(k+1)} . \qquad (5)$$

Where H, I and J are null matrices, and

$$E = \begin{bmatrix} 1 & \lambda_2 & \lambda_2 & . & . & . & \lambda_2 \\ & 3\lambda_4 & \lambda_4 & . & . & . & \lambda_4 \\ & & 3\lambda_4 & . & . & . & \lambda_4 \\ & & & . & & & . \\ & & & & . & & . \\ & & & & & . & . \\ & & & & & & 3\lambda_4 \end{bmatrix}_{(k+1)\times(k+1)} \quad (6)$$

$$F = \begin{bmatrix} \lambda_4 & 0 & . & . & . & 0 \\ & \lambda_4 & . & . & . & 0 \\ & & . & & & . \\ & & & . & & . \\ & & & & . & . \\ & & & & & \lambda_4 \end{bmatrix}_{k\times k} \quad (7)$$ and

$$G = \begin{bmatrix} \lambda_2 & 0 & . & . & . & 0 \\ & \lambda_2 & . & . & . & 0 \\ & & . & & & . \\ & & & . & & . \\ & & & & . & . \\ & & & & & \lambda_2 \end{bmatrix}_{k\times k} \quad (8)$$

That is $M_{(K+1)\times(K+1)} = \begin{bmatrix} E & 0 & 0 \\ & F & 0 \\ sym & & G \end{bmatrix}$ (9)

Their inverses and determinants are;

$$E^{-1} = \frac{1}{[2\lambda_4][k\lambda_2^{\ 2} - (k+2)\lambda_4]} \begin{bmatrix} -(k+2)\lambda_4^{\ 2} & 2\lambda_2\lambda_4 & 2\lambda_2\lambda_4 & . & . & 2\lambda_2\lambda_4 \\ & (k-1)\lambda_2^{\ 2} - (k+1)\lambda_4 & \lambda_4 - \lambda_2^{\ 2} & . & . & \lambda_4 - \lambda_2^{\ 2} \\ & & (k-1)\lambda_2^{\ 2} - (k+1)\lambda_4 & . & . & \lambda_4 - \lambda_2^{\ 2} \\ & & & . & . & . \\ & \textit{symmetric} & & . & . & . \\ & & & & & (k-1)\lambda_2^{\ 2} - (k+1)\lambda_4 \end{bmatrix}_{(k+1)\times(k+1)} \tag{10}$$

This gives,

$$|E| = (2\lambda_4)(k\lambda_2^{\ 2} - (k+2)\lambda_4) \tag{11}$$

$$F^{-1} = \begin{bmatrix} \lambda_4^{\ -1} & 0 & 0 & . & . & . & 0 \\ & \lambda_4^{\ -1} & 0 & . & . & . & \lambda_4 \\ & & \lambda_4^{\ -1} & . & . & . & \lambda_4 \\ & & & . & & & . \\ & & & & . & & . \\ \textit{Symmetric} & & & & & . & . \\ & & & & & & \lambda_4^{\ -1} \end{bmatrix}_{k\times k} \tag{12}$$

This gives $$|F| = \lambda_4 \tag{13}$$

and
$$G^{-1} = \begin{bmatrix} \lambda_2^{-1} & 0 & 0 & . & . & . & 0 \\ & \lambda_2^{-1} & 0 & . & . & . & 0 \\ & & \lambda_2^{-1} & . & . & . & 0 \\ & & & . & & & . \\ & & & & . & & . \\ \mathit{Symmetric} & & & & & . & . \\ & & & & & & \lambda_2^{-1} \end{bmatrix}_{k \times k} \tag{14}$$

This gives $$|G| = \lambda_2 \tag{15}$$

The determinant of $|M|$ was given as the product of $|E|\,|F|\,|G|$

Therefore,

$$|M| = \left(2\lambda_2\lambda_4{}^2\right)\left(k\lambda_2{}^2 - (k+2)\lambda_4\right) \tag{16}$$

From the equation (16),

$$\left(2\lambda_2\lambda_4{}^2\right) \neq 0 \quad \text{and} \quad \left(k\lambda_2{}^2 - (k+2)\lambda_4\right) \neq 0$$

Then non-singularity condition is obtained i.e. $\frac{\lambda_4}{\lambda_2{}^2} \neq \frac{k}{k+2}$ i.e $\frac{NB}{A^2} \geq \frac{k}{k+2}$ where $A = N\lambda_2$ and $B = N\lambda_4$

3.3.2 Review of Variances and Co-variances of the Parameter Estimates

The variances and co-variances are obtained by determining the inverses of the diagonal sub-matrices of the moment matrix M.

$$M^{-1} = \begin{bmatrix} E^{-1} & 0 & 0 \\ & F^{-1} & 0 \\ \mathit{Symmetric} & & G^{-1} \end{bmatrix}$$

In order to get the inverse of M, the inverses of the sub-matrices E, F and G were calculated. Considering the sub-matrices E let inverse of the matrix be

$$E_{(k\times k)} = \begin{bmatrix} a & b & b & . & . & . & b \\ & c & d & . & . & . & d \\ & & & . & & & . \\ & & & & . & & . \\ & & & & & . & . \\ & & & & & & c \end{bmatrix} \qquad (17)$$

$$\begin{bmatrix} 1 & \lambda_2 & \lambda_2 & . & . & . & \lambda_2 \\ & 3\lambda_4 & \lambda_4 & . & . & . & \lambda_4 \\ & & 3\lambda_4 & . & . & . & \lambda_4 \\ & & & . & & & . \\ & & & & . & & . \\ & Symmetric & & & & . & . \\ & & & & & & 3\lambda_4 \end{bmatrix} \begin{bmatrix} a & b & b & . & . & . & b \\ & c & d & . & . & . & d \\ & & c & . & . & . & d \\ & & & . & & & . \\ & & & & . & & . \\ & Symmetric & & & & . & . \\ & & & & & & d \end{bmatrix} = I_{k+1} \qquad (18)$$

$$\begin{aligned} & a + b\lambda_2 + b\lambda_2 + ... + b\lambda_2 = 1 && \text{(i)} \\ & a\lambda_2 + 3b\lambda_4 + b\lambda_4 + ... + b\lambda_4 = 0 && \text{(ii)} \qquad (19) \\ & b\lambda_2 + 3c\lambda_4 + d\lambda_4 + ... + d\lambda_4 = 1 && \text{(iii)} \\ & b\lambda_2 + c\lambda_2 + d\lambda_2 + ... + d\lambda_2 = 0 && \text{(iv)} \end{aligned}$$

That is,

$$\begin{aligned} & a + kb\lambda_2 = 1 && \text{(i)} \\ & a\lambda_2 + (k+2)b\lambda_4 = 0 && \text{(ii)} \\ & b\lambda_2 + 3c\lambda_4 + (k-1)d\lambda_4 = 1 && \text{(iii)} \qquad (20) \\ & b + c\lambda_2 + (k+2)d\lambda_4 = 0 && \text{(iv)} \end{aligned}$$

Solving the equations gives

From equation (20) (i) $\quad a = 1 - kb\lambda_2$

Substituting in equation (20) (ii)

$$(1 - kb\lambda_2)\lambda_2 + (k+2)b\lambda_4 = 0$$

And simplifying we obtain

$$b = \frac{\lambda_2}{k\lambda_2^{\ 2} - (k+2)\lambda_4} \tag{21}$$

Substituting the value of b, in equation 21(i) we obtain

$$a = \frac{-(k+2)\lambda_4}{k\lambda_2^{\ 2} - (k+2)\lambda_4} \tag{22}$$

From equation (iii) and (iv)

$$c = -\left(\frac{b + (k-1)d\lambda_2}{\lambda_2}\right)$$

and
$$c = \frac{1 - b\lambda_2 - (k-1)d\lambda_4}{3\lambda_4}$$

Solving the two equations above we get

$$d = \frac{b\lambda_2^{\ 2} - \lambda_2 - 3b\lambda_4}{2(k-1)\lambda_2\lambda_4}$$

And substituting for b

i.e.
$$b = \frac{\lambda_2}{k\lambda_2^{\ 2} - (k+2)\lambda_4}$$

Then,
$$d = \frac{\left[\frac{\lambda_2}{k\lambda_2^{\ 2} - (k+2)\lambda_4}\right]\lambda_2^{\ 2} - \lambda_2 - 3\lambda_4\left[\frac{\lambda_2}{k\lambda_2^{\ 2} - (k+2)\lambda_4}\right]}{2(k-1)\lambda_2\lambda_4}$$

Simplifying we get

$$d = \frac{(\lambda_4 - \lambda_2^{\ 2}}{2\lambda_4\left[k\lambda_2^{\ 2} - (k+2)\lambda_4\right]} \tag{23}$$

From

$$c = -\left(\frac{b + (k-1)d\lambda_2}{\lambda_2}\right)$$

and solving for c by substituting b and d

i.e.

$$c = -\frac{\left\{\dfrac{\lambda_2}{k\lambda_2^{\ 2} - (k+2)\lambda_4} + (k-1)\lambda_2\left[\dfrac{\left(\lambda_4 - \lambda_2^{\ 2}\right)}{2\lambda_4\left[k\lambda_2^{\ 2} - (k+2)\lambda_4\right]}\right]\right\}}{\lambda_2}$$

Solving the equation above, the value of c is given by

$$c = \frac{(k-1)\lambda_2^{\ 2} - (k+1)\lambda_4}{2\lambda_4\left[k\lambda_2^{\ 2} - (k+2)\lambda_4\right]} \tag{24}$$

The values of a, b, c and d obtained in equation (16) to (19) will give the inverse of E as

$$E^{-1} = \frac{1}{[2\lambda_4][k\lambda_2^{\ 2} - (k+2)\lambda_4]}\begin{bmatrix} -(k+2)\lambda_4^{\ 2} & 2\lambda_2\lambda_4 & 2\lambda_2\lambda_4 & . & . & 2\lambda_2\lambda_4 \\ & (k-1)\lambda_2^{\ 2} - (k+1)\lambda_4 & \lambda_4 - \lambda_2^{\ 2} & . & . & \lambda_4 - \lambda_2^{\ 2} \\ & & (k-1)\lambda_2^{\ 2} - (k+1)\lambda_4 & . & . & \lambda_4 - \lambda_2^{\ 2} \\ & & & . & . & . \\ & \mathit{Symmetric} & & . & . & . \\ & & & & & (k-1)\lambda_2^{\ 2} - (k+1)\lambda_4 \end{bmatrix}_{(k+1)\times(k+1)} \tag{25}$$

considering the submatrix $F = \begin{bmatrix} \lambda_4 & 0 & 0 & . & . & . & 0 \\ & \lambda_4 & 0 & . & . & . & 0 \\ & & \lambda_4 & . & . & . & 0 \\ & & & . & & & . \\ & & & & . & & . \\ & Symmetric & & & & . & . \\ & & & & & & \lambda_4 \end{bmatrix}_{k \times k}$

then the inverse of F is given as

$$F^{-1} = \begin{bmatrix} \lambda_4^{-1} & 0 & 0 & . & . & . & 0 \\ & \lambda_4^{-1} & 0 & . & . & . & 0 \\ & & \lambda_4^{-1} & . & . & . & 0 \\ & & & . & & & . \\ & & & & . & & . \\ & Symmetric & & & & . & . \\ & & & & & & \lambda_4^{-1} \end{bmatrix}_{k \times k} \qquad (26)$$

and

$$G = \begin{bmatrix} \lambda_2 & 0 & 0 & . & . & . & 0 \\ & \lambda_2 & 0 & . & . & . & 0 \\ & & \lambda_2 & . & . & . & 0 \\ & & & . & & & . \\ & & & & . & & . \\ & Symmetric & & & & . & . \\ & & & & & & \lambda_2 \end{bmatrix}_{k \times k}$$

the inverse of G is given as

$$G^{-1} = \begin{bmatrix} \lambda_2^{-1} & 0 & 0 & . & . & . & 0 \\ & \lambda_2^{-1} & 0 & . & . & . & 0 \\ & & \lambda_2^{-1} & . & . & . & 0 \\ & & & . & & & . \\ & & & & . & & . \\ & Symmetric & & & & . & . \\ & & & & & & \lambda_2^{-1} \end{bmatrix}_{k \times k} \quad (27)$$

therefore

$$M^{-1} = \begin{bmatrix} E^{-1} & 0 & 0 \\ & F^{-1} & 0 \\ Symmetric & & G^{-1} \end{bmatrix} \quad (28)$$

The variances and covariances of the second order polynomial were obtained by utilizing the inverse of the moment matrix .

Using the least squares estimation equation $X'X\underline{\beta} = X'\underline{Y}$, then if both sides is premultiplied by $(X'X)^{-1}$ we obtain

$(X'X)^{-1}\ X'X\underline{\beta} = (X'X)^{-1}\ X'\underline{Y}$

$\therefore \beta = (X'X)^{-1}X'\underline{Y}$

and the estimated variance is given by

$\text{var}(\hat{\beta}) = (X'X)^{-1}(X'X)(X'X)^{-1}\sigma^2 I_{k+1} = (X'X)^{-1}\sigma^2 I_{k+1}$

$= M^{-1}\sigma^2$

and from M^{-1} that was obtained from equation (28) the variances and covariances of the estimated response paremeters are given by

$$\text{var}(b_0) = \frac{-(k+2)\lambda_4^{\ 2}}{2\lambda_4[k\lambda_2^{\ 2} - (k+2)\lambda_4]}\sigma^2$$

$$\text{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\text{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\text{var}(b_{ii}) = \frac{(k-1)\lambda_2{}^2 - (k+1)\lambda_4}{2\lambda_4[k\lambda_2{}^2 - (k+2)\lambda_4]}\sigma^2$$

$$\text{cov}(b_0, b_{ii}) = \frac{\lambda_2\lambda_4}{\lambda_4[k\lambda_2{}^2 - (k+2)\lambda_4]}\sigma^2$$

$$\text{cov}(b_{ii}, b_{jj}) = \frac{\lambda_4 - \lambda_2{}^2}{2\lambda_4[k\lambda_2{}^2 - (k+2)\lambda_4]}\sigma^2$$

The variances and covariances for the modified conditions $\lambda_2{}^2 = \lambda_4$ gives

$$\text{var}(b_0) = \frac{(k+2)}{4}\sigma^2$$

$$\text{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\text{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\text{var}(b_{ii}) = \frac{1}{2\lambda_4}\sigma^2$$

$$\text{cov}(b_0, b_{ii}) = \frac{1}{2\lambda_2}\sigma^2$$

$$\text{cov}(b_{ii}, b_{jj}) = 0$$

CHAPTER FOUR

CONSTRUCTION OF SECOND ORDER ROTATABLE DESIGNS

4.1 Introduction

Second order rotatable designs in three, four, five and six and k- dimensions are constructed by considering a BIBD with the number of blocks equal to k, it is then added to the set of points $s(b,b,0....0)$ and $s(a,a,a....a)$ and a replicate of the incidence matrix obtained from BIBD. The variances and co variances were already obtained from the inverse of the moment matrix.

4.2 Construction Second Order Rotatable Designs in Three Dimensions

Considering the BIBD $(v=3, b=3, s=2, r=2, \lambda=1)$, then we have the design

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 3 & 1 & 0 \end{bmatrix} \tag{29}$$

Suppose we replace 1, 2 and 3 with x_{1u}, x_{2u} and 0 respectively in equation (29), then we obtain,

$$\begin{bmatrix} x_{1u} & x_{2u} & 0 \\ x_{2u} & 0 & 0 \\ 0 & x_{1u} & 0 \end{bmatrix} \tag{30}$$

Subjecting these equation to the non- singularity condition
We obtain

$$\sum_u x_{iu}^{\ 2} = x_{1u}^{\ 2} + x_{2u}^{\ 2} = 2A \tag{31}$$

$$\sum_u x_{iu}^{\ 4} = x_{1u}^{\ 4} + x_{2u}^{\ 4} = 2(3B) = 6B \tag{32}$$

and

$$\sum_u x_{iu}^{\ 2} x_{ju}^{\ 2} = x_{1u}^{\ 2} x_{2u}^{\ 2} = B \tag{33}$$

The set of points $s(b,b,0)$, $s(a,a,a)$ and a replicate of the incidence matrix was given as

$$\begin{bmatrix} b & b & 0 \\ b & -b & 0 \\ -b & b & 0 \\ -b & -b & 0 \\ 0 & b & b \\ 0 & -b & b \\ 0 & b & -b \\ 0 & -b & -b \\ b & 0 & b \\ -b & 0 & b \\ b & b & -b \\ -b & 0 & -b \end{bmatrix} \qquad (34)$$

$$\begin{bmatrix} a & a & a \\ a & a & -a \\ a & -a & a \\ -a & a & a \\ a & -a & -a \\ -a & -a & a \\ -a & a & -a \\ -a & -a & -a \end{bmatrix} \qquad (35)$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & b & -1 \\ -1 & 0 & -1 \end{bmatrix} \qquad (36)$$

From equations (34), (35) and (36) we have

$$\sum x_{iu}{}^4 = 8b^4 + 8a^4 + 8 \tag{37}$$

and

$$\sum x_{iu}{}^2 x_{ju}{}^2 = 4b^4 + 8a^4 + 4 \tag{38}$$

Combining equations (32) and (37) we have

$$2\sum x_{iu}{}^4 = 6B + 8b^4 + 8a^4 + 8 \tag{39}$$

This simplifies to

$$\sum x_{iu}{}^4 = 3B + 4b^4 + 4a^4 + 4$$

And Combining equations (33) and (38) we have

$$2\sum x_{iu}{}^2 x_{ju}{}^2 = B + 4b^4 + 8a^4 + 4 \tag{40}$$

Thus $\sum x_{iu}{}^4 - 3\sum x_{iu}{}^2 x_{ju}{}^2 = 0$ is given as

$$\left(3B + 4b^4 + 4a^4 + 4\right) - 3\left(\frac{B}{2} + 2b^4 + 4a^4 + 2\right) = 0$$

$$\left(3B + 4b^4 + 4a^4 + 4\right) - \left(\frac{3}{2}B + 6b^4 + 12a^4 + 6\right) = 0$$

$$\Rightarrow \frac{3}{2}B - 2b^4 - 8a^4 - 2 = 0$$

or

$$3B = 4b^4 + 16a^4 + 4 \tag{41}$$

Putting $N = 8$ in equation (2) then $B = \frac{8}{15}$ and equation (41) becomes

$$3 \times \frac{8}{15} = 4b^4 + 16a^4 + 4$$

That is $\frac{8}{5} = 4b^4 + 16a^4 + 4$ or $8 = 20b^4 + 80a^4 + 20$

Which gives $b^4 = \frac{-80a^4 - 12}{20}$ and therefore $-80a^4 - 12 \geq 0$

Thus $a^4 \geq -\frac{3}{20}$

The variances and co-variances of the parameter estimates of this design where $k = 3$ are given by

$$\text{var}(b_0) = \frac{5\lambda_4}{2[5\lambda_4 - 3\lambda_2^{\ 2}]}\sigma^2$$

$$\text{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\text{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\text{var}(b_{ii}) = \frac{\lambda_2^{\ 2} - 2\lambda_4}{\lambda_4[3\lambda_2^{\ 2} - 5\lambda_4]}\sigma^2$$

$$\text{cov}(b_0, b_{ii}) = \frac{\lambda_2}{[3\lambda_2^{\ 2} - 5\lambda_4]}\sigma^2$$

$$\text{cov}(b_{ii}, b_{jj}) = \frac{\lambda_4 - \lambda_2^{\ 2}}{2\lambda_4[3\lambda_2^{\ 2} - 5\lambda_4]}\sigma^2$$

applying the modified conditions $\lambda_2^{\ 2} = \lambda_4$ gives

$$\text{var}(b_0) = \frac{5}{4}\sigma^2$$

$$\text{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\text{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\text{var}(b_{ii}) = \frac{1}{2\lambda_4}\sigma^2$$

$$\text{cov}(b_0, b_{ii}) = \frac{1}{2\lambda_2}\sigma^2$$

$$\text{cov}(b_{ii}, b_{jj}) = 0$$

4.3 Construction of Second Order Rotatable Resigns in Four Dimensions

Considering the BIBD $(v = 4, b = 6, s = 2, r = 3, \lambda = 1)$ we have,

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix} \qquad (42)$$

Replacing 1, 2, 3 and 4 in (42) with x_{1u}, x_{2u}, 0 and 0 respectively, we have

$$\begin{bmatrix} x_{1u} & x_{2u} & 0 & 0 \\ x_{2u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x_{1u} & 0 & 0 \\ 0 & x_{1u} & 0 & 0 \\ x_{1u} & 0 & 0 & 0 \end{bmatrix} \qquad (43)$$

Subjecting these to rotatability condition we have

$$\sum_u x_{iu}{}^2 = x_{1u}{}^2 + x_{2u}{}^2 + x_{1u}{}^2 = 3A \qquad (44)$$

$$\sum_u x_{iu}{}^4 = x_{1u}{}^4 + x_{2u}{}^4 + x_{1u}{}^4 = 3(3B) = 9B \qquad (45)$$

$$\sum_u x_{iu}{}^2 x_{ju}{}^2 = B \qquad (46)$$

the set of points $s(b,b,0,0), s(a,a,a,a)$ and a replicate of the incidence matrix was be given by

$$\begin{bmatrix} b & b & 0 & 0 \\ b & -b & 0 & 0 \\ -b & b & 0 & 0 \\ -b & -b & 0 & 0 \\ b & 0 & b & 0 \\ b & 0 & -b & 0 \\ -b & 0 & b & 0 \\ -b & 0 & -b & 0 \\ b & 0 & 0 & b \\ b & 0 & 0 & -b \\ -b & 0 & 0 & b \\ -b & 0 & 0 & -b \\ 0 & b & b & 0 \\ 0 & b & -b & 0 \\ 0 & -b & b & 0 \\ 0 & -b & -b & 0 \\ 0 & b & 0 & b \\ 0 & b & 0 & -b \\ 0 & -b & 0 & b \\ 0 & -b & 0 & -b \\ 0 & 0 & b & b \\ 0 & 0 & b & -b \\ 0 & 0 & -b & b \\ 0 & 0 & -b & -b \end{bmatrix} \tag{47}$$

$$\begin{bmatrix} a & a & a & a \\ a & a & a & -a \\ a & a & -a & a \\ a & -a & a & a \\ -a & a & a & a \\ a & a & -a & -a \\ a & -a & -a & a \\ -a & -a & a & a \\ a & -a & a & -a \\ -a & a & -a & a \\ -a & a & a & -a \\ a & -a & -a & -a \\ -a & a & -a & -a \\ -a & -a & a & -a \\ -a & -a & -a & a \\ -a & -a & -a & -a \end{bmatrix} \qquad (48)$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \qquad (49)$$

which gives,

$$\sum x_{iu}{}^{4} = 12b^{4} + 16a^{4} + 12 \qquad (50)$$

$$\sum x_{iu}{}^{2} x_{ju}{}^{2} = 4b^{4} + 16a^{4} + 4 \qquad (51)$$

Combining equation (45) and (50) we have

$$2\sum x_{iu}{}^{4} = 9B + 12b^{4} + 16a^{4} + 12 \qquad (52)$$

And equation (46) and (51) gives

$$2\sum x_{iu}^{\ 2} x_{ju}^{\ 2} = B + 12b^4 + 16a^4 + 4 \tag{53}$$

and $\sum x_{iu}^{\ 4} - 3\sum x_{iu}^{\ 2} x_{ju}^{\ 2} = 0$ gives us

$$\left(9B + 12b^4 + 16a^4 + 12\right) - 3\left(B + 4b^4 + 16a^4 + 4\right) = 0$$

$$9B + 12b^4 + 16a^4 + 12 - \left(3B + 12b^4 + 48a^4 + 12\right) = 0$$

$$9B + 12b^4 + 16a^4 + 12 - 3B - 12b^4 - 48a^4 - 12 = 0$$

$$\Rightarrow 6B - 32a^4 = 0$$

or

$$6B = 32a^4 \tag{54}$$

Putting $N = 8$ in equation (2) then $B = \dfrac{8}{24}$ and equation (54) becomes

$a^4 = \dfrac{1}{16}$ and $a = \pm\dfrac{1}{2}$

In this case the value of a is defined and that of b is not defined.

The variances and co-variances of the parameter estimates of this design where are given as

$$\text{var}(b_0) = \frac{3\lambda_4}{2[3\lambda_4 - 2\lambda_2^{\ 2}]}\sigma^2$$

$$\text{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\text{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\text{var}(b_{ii}) = \frac{3\lambda_2^{\ 2} - 5\lambda_4}{4\lambda_4[2\lambda_2^{\ 2} - 3\lambda_4]}\sigma^2$$

$$\text{cov}(b_0, b_{ii}) = \frac{\lambda_2}{2[2\lambda_2^{\ 2} - 3\lambda_4]}\sigma^2$$

$$\text{cov}(b_{ii}, b_{jj}) = \frac{\lambda_4 - \lambda_2^{\ 2}}{4\lambda_4[2\lambda_2^{\ 2} - 3\lambda_4]}\sigma^2$$

and applying the modified condition $\lambda_2^{\ 2} \quad \lambda_4$ and the variances and covariances become

$$\text{var}(b_0) = \frac{3}{4}\sigma^2$$

$$\text{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\text{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\text{var}(b_{ii}) = \frac{1}{2\lambda_4}\sigma^2$$

$$\text{cov}(b_0, b_{ii}) = \frac{1}{2\lambda_2}\sigma^2$$

$$\text{cov}(b_{ii}, b_{jj}) = 0$$

4.4 Construction of Second Order Rotatable Design in Five Dimensions

Considering the BIB $(v = 5, b = 10, s = 2, r = 4, \lambda = 1)$ then we have

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 & 0 \end{bmatrix} \qquad (55)$$

Replacing 1, 2, 3, 4 and 5 with $x_{1u}, x_{2u}, 0, 0$ and 0 respectively, in equation (55), we have,

$$\begin{bmatrix} x_{1u} & x_{2u} & 0 & 0 & 0 \\ x_{2u} & 0 & 0 & 0 & 0 \\ 0 & x_{1u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_{1u} & 0 & 0 & 0 & 0 \\ x_{2u} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & x_{1u} & 0 & 0 & 0 \\ 0 & x_{2u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (56)$$

Subjecting equation (56) to the rotatability condition, we obtain

$$\sum_u x_{iu}{}^2 = x_{1u}{}^2 + x_{2u}{}^2 + x_{1u}{}^2 + x_{2u}{}^2 = 4A \quad (57)$$

$$\sum_u x_{iu}{}^4 = x_{1u}{}^4 + x_{2u}{}^4 + x_{1u}{}^4 + x_{2u}{}^4 = 4(3B) = 12B \quad (58)$$

$$\sum_u x_{iu}{}^2 x_{ju}{}^2 = B \quad (59)$$

The set of points $s(b,b,0,0,0)$, $s(a,a,a,a,a)$ and a replicate of the incidence matrix will be given by

$$\begin{bmatrix} b & b & 0 & 0 & 0 \\ b & -b & 0 & 0 & 0 \\ -b & b & 0 & 0 & 0 \\ -b & -b & 0 & 0 & 0 \\ b & 0 & b & 0 & 0 \\ b & 0 & -b & 0 & 0 \\ -b & 0 & b & 0 & 0 \\ -b & 0 & -b & 0 & 0 \\ b & 0 & 0 & b & 0 \\ b & 0 & 0 & -b & 0 \\ -b & 0 & 0 & b & 0 \\ -b & 0 & 0 & -b & 0 \\ b & 0 & 0 & 0 & b \\ b & 0 & 0 & 0 & -b \\ -b & 0 & 0 & 0 & b \\ -b & 0 & 0 & 0 & -b \\ 0 & b & b & 0 & 0 \\ 0 & b & -b & 0 & 0 \\ 0 & -b & b & 0 & 0 \\ 0 & -b & -b & 0 & 0 \\ 0 & b & 0 & b & 0 \\ 0 & b & 0 & -b & 0 \\ 0 & -b & 0 & b & 0 \\ 0 & -b & 0 & -b & 0 \\ 0 & b & 0 & 0 & b \\ 0 & b & 0 & 0 & -b \\ 0 & -b & 0 & 0 & b \\ 0 & -b & 0 & 0 & -b \\ 0 & 0 & b & b & 0 \\ 0 & 0 & b & -b & 0 \\ 0 & 0 & -b & b & 0 \\ 0 & 0 & -b & -b & 0 \\ 0 & 0 & b & 0 & b \\ 0 & 0 & b & 0 & -b \\ 0 & 0 & -b & 0 & b \\ 0 & 0 & -b & 0 & -b \\ 0 & 0 & 0 & b & b \\ 0 & 0 & 0 & b & -b \\ 0 & 0 & 0 & -b & b \\ 0 & 0 & 0 & -b & -b \end{bmatrix} \tag{60}$$

$$\begin{bmatrix} a & a & a & a & a \\ a & a & a & a & -a \\ a & a & a & -a & a \\ a & a & -a & a & a \\ a & -a & a & a & a \\ -a & a & a & a & a \\ a & -a & a & a & -a \\ -a & a & a & a & -a \\ -a & a & a & -a & a \\ -a & a & -a & a & a \\ -a & -a & a & a & a \\ a & -a & a & a & -a \\ a & -a & a & -a & a \\ a & -a & -a & a & a \\ a & a & -a & a & -a \\ a & a & -a & -a & a \\ a & a & a & -a & -a \\ -a & a & a & -a & -a \\ -a & a & -a & -a & a \\ -a & -a & -a & a & a \\ -a & a & -a & a & -a \\ -a & -a & a & a & -a \\ -a & -a & a & -a & a \\ a & -a & a & -a & -a \\ a & -a & -a & -a & a \\ a & a & -a & -a & -a \\ -a & -a & -a & -a & a \\ -a & -a & -a & a & -a \\ -a & -a & a & -a & -a \\ -a & a & -a & -a & -a \\ a & -a & -a & -a & -a \\ -a & -a & -a & -a & -a \end{bmatrix} \tag{61}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \tag{62}$$

From (60) to (62) we have

$$\sum x_{iu}^{\ 4} = 16b^4 + 32a^4 + 16 \tag{63}$$

$$\sum x_{iu}^{\ 2} x_{ju}^{\ 2} = 4b^4 + 32a^4 + 4 \tag{64}$$

Combining (58) and (63) we have

$$2\sum x_{iu}^{\ 4} = 12B + 16b^4 + 32a^4 + 16 \tag{65}$$

And combining (59) and (64) we get

$$2\sum x_{iu}^{\ 2} x_{ju}^{\ 2} = B + 4b^4 + 32a^4 + 4 \tag{66}$$

Subjecting equation (65) and (66) to the condition

$$\sum x_{iu}^{\ 2} - 3\sum x_{iu}^{\ 2} x_{ju}^{\ 2} = 0\,,$$

we obtain

$$(12B + 16b^4 + 32a^4 + 16) - 3(B + 4b^4 + 32a^4 + 4) = 0$$

$$12B + 16b^4 + 32a^4 + 16 - (3B + 12b^4 + 96a^4 + 12) = 0$$

$$\Rightarrow 9B + 4b^4 - 64a^4 + 4 = 0$$

$$9B = 64a^4 - 4b^4 - 4 \tag{67}$$

Putting $N = 8$, in equation (2) then $B = \frac{8}{35}$ and equation (67) becomes

$$9 \times \frac{8}{35} = 64a^4 - 4b^4 - 4$$

$$\frac{72}{35} = 64a^4 - 4b^4 - 4$$

$$\frac{212}{35} = 64a^4 - 4b^4$$

$$64a^4 = \frac{212}{35} + 4b^4$$

Therefore $4b^4 = 64a^4 - \frac{212}{35}$ or

$b^4 = \frac{560a^4 - 53}{35}$ and $a^4 \geq \frac{53}{560}$

The variances and co-variances of the parameter estimates of this design where $k = 5$ are given as

$$\mathrm{var}(b_0) = \frac{7\lambda_4}{2[7\lambda_4 - 5\lambda_2{}^2]}\sigma^2$$

$$\mathrm{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\mathrm{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\mathrm{var}(b_{ii}) = \frac{2\lambda_2{}^2 - 3\lambda_4}{\lambda_4[5\lambda_2{}^2 - 7\lambda_4]}\sigma^2$$

$$\mathrm{cov}(b_0, b_{ii}) = \frac{\lambda_2}{[5\lambda_2{}^2 - 7\lambda_4]}\sigma^2$$

$$\mathrm{cov}(b_{ii}, b_{jj}) = \frac{\lambda_4 - \lambda_2{}^2}{2\lambda_4[5\lambda_2{}^2 - 7\lambda_4]}\sigma^2$$

and applying the modified condition $\lambda_2{}^2 = \lambda_4$ we obtain the variances and covariances as

$$\mathrm{var}(b_0) = \frac{7}{4}\sigma^2$$

$$\mathrm{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\mathrm{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\mathrm{var}(b_{ii}) = \frac{1}{2\lambda_4}\sigma^2$$

$$\mathrm{cov}(b_0, b_{ii}) = \frac{1}{2\lambda_2}\sigma^2$$

$$\mathrm{cov}(b_{ii}, b_{jj}) = 0$$

4.5 Construction of Second Order Rotatable Design in Six Dimensions

Consider the BIB $(v=6, b=15, s=2, r=5, \lambda=1)$ then we have

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (68)$$

Replacing 1, 2, 3, 4, 5 and 6 in (68) with $x_{1u} x_{2u}, 0,0,0$ and 0 respectively

$$\begin{bmatrix} x_{1u} & x_{2u} & 0 & 0 & 0 & 0 \\ x_{2u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{1u} & 0 & 0 & 0 & 0 \\ x_{1u} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{2u} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{1u} & 0 & 0 & 0 & 0 \\ x_{2u} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{2u} & 0 & 0 & 0 & 0 \\ x_{1u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (69)$$

Subjecting equation (69) to the rotatability condition, we obtain

$$\sum_u x_{iu}{}^2 = x_{1u}{}^2 + x_{2u}{}^2 + x_{1u}{}^2 + x_{1u}{}^2 + x_{2u}{}^2 = 5A \qquad (70)$$

$$\sum_u x_{iu}^{\ 4} = x_{1u}^{\ 4} + x_{2u}^{\ 4} + x_{1u}^{\ 4} + x_{2u}^{\ 4} + x_{1u}^{\ 4} = 5(3B) = 15B \qquad (71)$$

$$\sum_u x_{iu}^{\ 2} x_{ju}^{\ 2} = B \qquad (72)$$

From the sum of the set of points

$s(b,b,0,0,0,0) + s(a,a,a,a,a,a) +$ the incidence matrix

give rise to

$$\sum x_{iu}^{\ 4} = 20b^4 + 64a^4 + 20 \qquad (73)$$

and

$$\sum x_{iu}^{\ 2} x_{ju}^{\ 2} = 4b^4 + 64a^4 + 4, \qquad (74)$$

Combining (71) and (73) we have

$$2\sum x_{iu}^{\ 4} = 15B + 20b^4 + 64a^4 + 20$$

And combining (72) and (74) we have

$$2\sum x_{iu}^{\ 2} x_{ju}^{\ 2} = B + 4b^4 + 64a^4 + 4$$

Therefore,

$$\sum x_{iu}^{\ 4} - 3\sum x_{iu}^{\ 2} x_{ju}^{\ 2} = 0$$

gives us

$$\left(15B + 20b^4 + 64a^4 + 20\right) - 3\left(B + 4b^4 + 64a^4 + 4\right) = 0$$

$$\left(15B + 20b^4 + 64a^4 + 20\right) - \left(3B + 12b^4 + 192a^4 + 12\right) = 0$$

$$\Rightarrow 12B + 8b^4 - 128a^4 + 8 = 0$$

or

$$12B = 128a^4 - 8b^4 - 8 \qquad (75)$$

Putting $N = 8$ in equation (2), then $B = \frac{1}{6}$ equation (75) becomes

$$\frac{12}{6} = 128a^4 - 8b^4 - 8$$

That is $128a^4 - 8b^4 = 10$ or $64a^4 - 4b^4 = 5$

Therefore $b^4 = \frac{64a^4 - 5}{4}$ and $a^4 \geq \frac{5}{64}$

The variances and co-variances of the parameter estimates of this design where $k = 6$ are given as

$$\text{var}(b_0) = \frac{2\lambda_4}{[4\lambda_4 - 3\lambda_2{}^2]}\sigma^2$$

$$\text{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\text{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\text{var}(b_{ii}) = \frac{5\lambda_2{}^2 - 7\lambda_4}{4\lambda_4[3\lambda_2{}^2 - 4\lambda_4]}\sigma^2$$

$$\text{cov}(b_0, b_{ii}) = \frac{\lambda_2}{2[3\lambda_2{}^2 - 4\lambda_4]}\sigma^2$$

$$\text{cov}(b_{ii}, b_{jj}) = \frac{\lambda_4 - \lambda_2{}^2}{4\lambda_4[3\lambda_2{}^2 - 4\lambda_4]}\sigma^2$$

and applying the modified condition $\lambda_2{}^2 = \lambda_4$ we obtain the variances and covariances as

$$\text{var}(b_0) = 2\sigma^2$$

$$\text{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\text{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\text{var}(b_{ii}) = \frac{1}{2\lambda_4}\sigma^2$$

$$\text{cov}(b_0, b_{ii}) = \frac{1}{2\lambda_2}\sigma^2$$

$$\text{cov}(b_{ii}, b_{jj}) = 0$$

4.6 Construction of Second Order Rotatable Designs in k −dimensions $(k > 6)$

The design in k dimensions is obtained by the generalization of the design in three, four, five and six factors. The BIBD considered is

$(v = k, b, r = k - 1, s = 2, \lambda = 1)$

The sum of the set of points $s(b, b, 0, 0 \ldots, 0) + s(a, a, a, \ldots a) +$ a replicate of the incidence matrix gave rise to

$$\sum_u x_{iu}^{\ 2} = (k-1)A \tag{76}$$

$$2\sum x_{iu}^{\ 4} = 3(k-1)B + 4(k-1)b^4 + 2^k a^4 + 4(k-1) \tag{77}$$

and

$$2\sum x_{iu}^{\ 2} x_{ju}^{\ 2} = B + 4b^4 + 2^k a^a + 4 \tag{78}$$

Subjecting equation (77) and (78) to the condition

$$\sum x_{iu}^{\ 4} - 3\sum x_{iu}^{\ 2} x_{ju}^{\ 2} = 0$$

gives,

$$3(k-1)B + 4(k-1)b^4 + 2^k a^4 + 4(k-1) - 3(B + 4b^4 + 2^k a^4 + 4) = 0$$

$$3(k-1)B + 4(k-1)b^4 + 2^k a^4 + 4(k-1) - 3B - 12b^4 3\left(2^k a^4\right) + 12 = 0$$

$$(3k-6)B + (4k-16)b^4 + 2\left(2^k a^4\right) + (4k-16) = 0$$

Which simplifies to

$$3(k-2)B + 4(k-4)b^4 - 2^{k+1} a^4 + 4(k-4) = 0 \tag{79}$$

The variances and co variances of this design are given as

$$\text{var}(b_0) = \frac{(k+2)\lambda_4^{\ 2}}{2\lambda_4[\left(k+2\right)\lambda_4 - k\lambda_2^{\ 2}]}\sigma^2$$

$$\text{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\text{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\text{var}(b_{ii}) = \frac{(k-1)\lambda_2^{\ 2} - (k+1)\lambda_4}{2\lambda_4[k\lambda_2^{\ 2} - (k+2)\lambda_4]}\sigma^2$$

$$\text{cov}(b_0, b_{ii}) = \frac{\lambda_2\lambda_4}{\lambda_4[k\lambda_2^{\ 2} - (k+2)\lambda_4]}\sigma^2$$

$$\text{cov}(b_{ii}, b_{jj}) = \frac{\lambda_4 - \lambda_2^{\ 2}}{2\lambda_4[k\lambda_2^{\ 2} - (k+2)\lambda_4]}\sigma^2$$

The variances and covariances for the modified conditions $\lambda_2^{\ 2} = \lambda_4$ gives

$$\text{var}(b_0) = \frac{(k+2)}{4}\sigma^2$$

$$\mathrm{var}(b_i) = \frac{1}{\lambda_2}\sigma^2$$

$$\mathrm{var}(b_{ij}) = \frac{1}{\lambda_4}\sigma^2$$

$$\mathrm{var}(b_{ii}) = \frac{1}{2\lambda_4}\sigma^2$$

$$\mathrm{cov}(b_0, b_{ii}) = \frac{1}{2\lambda_2}\sigma^2$$

$$\mathrm{cov}(b_{ii}, b_{jj}) = 0$$

CHAPTER FIVE

OPTIMALITY CRITERIA

5.1 Trace Criterion

Considering the moment matrix M, the trace of a matrix is given as the sum of all the elements in the principal diagonal, thus

$$\frac{1}{k+1}tr(M) = \frac{1}{k+1}\sum_{i=1}^{k+1} M_{ii}$$

$$= \frac{1}{k+1}\left[(k+1) + 3\lambda_4 k + \lambda_4 k + \lambda_2 k\right]$$

$$= \frac{1}{k+1}\left[(k+1) + 4\lambda_4 k + \lambda_2 k\right]$$

$$= 1 + \frac{1}{k+1}\left[(4\lambda_4 k + \lambda_2 k\right]$$

5.2 Determinant

The determinant of the matrix was obtained in equation (16) as

$$|M| = 2\lambda_2 \lambda_4^2 \left(k\lambda_2^2 - (k+2)\lambda_4\right)$$

5.3 Eigen Values

The Eigen value of the matrix M is given by

$$|\lambda I_{k+1} - M| = 0$$

Which produces a characteristic polynomial of degree $k+1$ the $(k+1)\lambda$ is given as $\det(\lambda I_{k+1} - M)$

Considering the sub-matrix $E_{k\times k}$ then the Eigen value can be obtained as

$$|\lambda I_{(k+1)\times(k+1)} - E| = \begin{bmatrix} \lambda - 1 & -\lambda_2 & -\lambda_2 & . & . & . & -\lambda_2 \\ & \lambda - 3\lambda_4 & -\lambda_4 & . & . & . & -\lambda_4 \\ & & \lambda - 3\lambda_4 & . & . & . & \lambda_4 \\ & & & . & & & . \\ & & & & . & & . \\ Symmetric & & & & & . & . \\ & & & & & & 3\lambda_4 \end{bmatrix}_{(k+1)\times(k+1)} \qquad (80)$$

the determinant of E was already obtained in equation (11), therefore

$$\left(\lambda - 3\lambda_2{}^2 + (k+2)\lambda_4\right)\left[\lambda - 2\lambda_4\right]^k = 0$$

Thus $\lambda = 3\lambda_2{}^2 - (k+2)\lambda_4$

Or $\lambda = 2\lambda_4$

for sub-matrix F

$$\left|\lambda I_k - F\right| = 0$$

$$\begin{bmatrix} \lambda - 1 & 0 & 0 & . & . & . & 0 \\ & \lambda - \lambda_4 & 0 & . & . & . & 0 \\ & & \lambda - \lambda_4 & . & . & . & 0 \\ & & & . & & & . \\ & & & & . & & . \\ \textit{Symmetric} & & & & & . & . \\ & & & & & & \lambda - \lambda_4 \end{bmatrix}_{k \times k} = 0 \qquad (81)$$

$$\left(\lambda - \lambda_4\right)^k = 0$$

$$\therefore \lambda = \lambda_4 \qquad (82)$$

and for G

$\left|\lambda I_k - G\right| = 0$ gives

$$\begin{bmatrix} \lambda - 1 & 0 & 0 & . & . & . & 0 \\ & \lambda - \lambda_2 & 0 & . & . & . & 0 \\ & & \lambda - \lambda_2 & . & . & . & 0 \\ & & & . & & & . \\ & & & & . & & . \\ \textit{Symmetric} & & & & & . & . \\ & & & & & & \lambda - \lambda_2 \end{bmatrix}_{k \times k} = 0 \qquad (83)$$

$$\left(\lambda - \lambda_2\right)^k = 0$$

$$\therefore \lambda = \lambda_2 \qquad (84)$$

CHAPTER SIX

CONCLUSION AND RECOMMENDATION

6.1 Conclusion

New designs generated by BIBD were obtained in three, four, five, six and a generalization in k- dimension which was given in chapter 4.The general form of these equations were given by $s(b,b,0,0...,0)+s(a,a,a,...a)+$ a replicate of the incidence matrix of BIB. Their variances and co-variances were obtained respectively. The variances and co-variances under the modified conditions $\lambda_4=\lambda_2{}^2$ were also reviewed and were similar except the $\text{var}(b_0)$ and the $\text{cov}(b_{ii}b_{ij})$ which were zero in all the designs.

Finally the optimality criteria were also obtained that is the trace, eigen values and determinant.

It is seen that if $\lambda_4=\lambda_2{}^2$ the variances and co-variances of the parameter estimates becomes similar except the $\text{var}(b_0)$ and the $\text{cov}(b_{ii}b_{ij})=0$ .Therefore the modification of the variances and co-variances affect the estimated response.

If an experimenter might be interested in some k subset of factors then the results in these studies will be desirable since this subset will be identified with blocks generating a BIBD (k, b, s, r, λ) so that the second order k-dimensional design contain second order rotatable designs in k-1 dimensions involving the subsets of factors the experimenter is interested in. where the experimenter is interested in replicate, then the design is suitable since a replicate of the incidence matrix of BIB is used.

The experimental designs that were obtained in this study ensure equal precision on the response to cut down on cost. The design can be useful in agriculture, textile industry, motor vehicle industry and all other types of industry that make use of experimental designs to manufacture their products.

6.2 Recommendation

A generalization of second order rotatable designs based on balanced incomplete block designed was determined in this study. Another area in which one may be interested in is the generalization of third order designs into k-dimensions.

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